Electronic Journal of Qualitative Theory of Differential Equations
2023, No. **41**, 1–10; https://doi.org/10.14232/ejqtde.2023.1.41 www.math.u-szeged.hu/ejqtde/

Qualitative analysis on the diffusive Holling–Tanner predator–prey model

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Received 6 November 2022, appeared 26 August 2023
Communicated by Sergei Trofimchuk

Abstract. We consider the diffusive Holling–Tanner predator–prey model subject to the homogeneous Neumann boundary condition. We first apply Lyapunov function method to prove some global stability results of the unique positive constant steady-state. And then, we derive a non-existence result of positive non-constant steady-states by a novel approach that can also be applied to the classical Sel'kov model to obtain the non-existence of positive non-constant steady-states if $0 < p \leq 1$.

Keywords: Holling–Tanner predator–prey model, Sel'kov model, global stability, Lyapunov function method.

2020 Mathematics Subject Classification: 34C20, 34C25, 92E20.

1 Introduction

In this paper, we consider the diffusive Holling–Tanner predator–prey model:

$$\begin{cases} u_t - d_1\Delta u = au - u^2 - \dfrac{uv}{m+u}, & x \in \Omega, t > 0, \\ v_t - d_2\Delta v = bv - \dfrac{v^2}{\gamma u}, & x \in \Omega, t > 0, \\ \partial_\nu u = \partial_\nu v = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x) > 0,\ v(x,0) = v_0(x) > 0, & x \in \overline{\Omega}. \end{cases} \tag{1.1}$$

Here u and v are the density of prey and predator, respectively, $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, ν is the outward unit normal vector on $\partial\Omega$, and the parameters $d_1, d_2, a, b, m, \gamma$ are positive constants. The initial data u_0 and v_0 are $C^1(\overline{\Omega})$ functions satisfying $\partial_\nu u_0 = \partial_\nu v_0 = 0$ on $\partial\Omega$. The model describes real ecological interactions of various populations such as lynx and hare, sparrow and sparrow hawk (cf. [7, 13, 15]), and the Neumann boundary condition means that no species can pass across the boundary $\partial\Omega$. We note that problem (1.1) has a unique positive global solution, see the Appendix for the proof.

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It is easy to verify that system (1.1) has a unique positive equilibrium $\mathbf{E}_* = (u_*, v_*)$, where

$$u_* = \frac{1}{2}\left[a - m - b\gamma + \sqrt{(a-m-b\gamma)^2 + 4am}\right] \quad \text{and} \quad v_* = b\gamma u_*.$$

System (1.1) had been extensively investigated, see [1, 2, 4, 5, 9–11] and the references therein. In particular, Peng and Wang [10], Chen and Shi [1], Duan, Niu and Wei [2], and Qi and Zhu [11] proved some stability results that are collected as follows.

Theorem 1.1. *Suppose $d_1, d_2, a, m, b, \gamma$ are positive constants. Then the following statements hold.*

(a) (See [10]). The positive equilibrium $\mathbf{E}_*$ is locally asymptotically stable if $m^2 + 2(a+b\gamma)m + a^2 - 2ab\gamma > 0$.

(b) (See [1]). The positive equilibrium $\mathbf{E}_*$ is globally asymptotically stable if $m > b\gamma$.

(c) (See [2]). The positive equilibrium $\mathbf{E}_*$ is globally asymptotically stable if $u_* \geq m$ and $m \geq a - u_*$.

(d) (See [11]). $\lim_{t\to+\infty}(u(x,t), v(x,t)) = \mathbf{E}_*$ uniformly on $\overline{\Omega}$ if $d_1 = d_2$ and $\gamma^{-1} > \frac{a}{m+a}$.

Motivated by the above works in [1,2,10,11], in the present paper, we first study the global stability of the positive equilibrium $\mathbf{E}_*$, and obtain the following result.

Theorem 1.2. *Suppose $d_1, d_2, a, m, b, \gamma$ are positive constants. Then the positive equilibrium $\mathbf{E}_*$ is globally asymptotically stable if $m > \max\{M_1, M_2\}$, where*

$$M_1 = \frac{ab\gamma}{a + b\gamma} \quad \text{and} \quad M_2 = \frac{1}{2}\left[(b\gamma - 2a)_+ + \sqrt{b\gamma(b\gamma - 2a)_+}\right].$$

Here $s_+ = \max\{0, s\}$.

Obviously, $M_1, M_2 < b\gamma$. Then Theorem 1.2 is an improvement to Theorem 1.1(b). Since $a - u_* = \frac{v_*}{m+u_*} = \frac{b\gamma u_*}{m+u_*}$, we see that $a - u_* < m \Leftrightarrow b\gamma u_* < m(m+u_*)$, so $a - u_* < m \Leftrightarrow m > M_1$ according to Lemma 2.1(a). On the other hand, since the condition $m \leq u_*$ implies $\frac{am}{a+2m} < u_*$, it also implies $m > M_2$ because $\frac{am}{a+2m} < u_* \Leftrightarrow m > M_2$ according to Lemma 2.1(b). Thus, Theorem 1.2 is also an improvement to Theorem 1.1(c).

Note that for fixed a, b and m, every global result in Theorems 1.1 and 1.2 excludes the case where γ is large. In this paper, we prove the following result that covers the case.

Theorem 1.3. *Suppose $d_1, d_2, a, m, b, \gamma$ are positive constants with $d_1 = d_2, b > a, m > M_1 = \frac{ab\gamma}{a+b\gamma}$, and*

$$2am\left[a + 2m + \frac{2m(b-a)}{m+a}\gamma\right]^{-1} < a - m - b\gamma + \sqrt{(a-m-b\gamma)^2 + 4am}. \tag{1.2}$$

Then the positive equilibrium $\mathbf{E}_*$ is globally asymptotically stable.

Remark 1.4. Let $m > a$ and $b > \frac{2am}{m-a}$. Then $m > M_1$ and (1.2) hold for any sufficiently large γ. Indeed, we have

$$\begin{aligned}
&\lim_{\gamma\to+\infty} \gamma\left[a - m - b\gamma + \sqrt{(a-m-b\gamma)^2 + 4am} - \frac{2am}{a + 2m + \frac{2m(b-a)}{m+a}\gamma}\right] \\
&= \lim_{\gamma\to+\infty}\left[\frac{4am\gamma}{\sqrt{(a-m-b\gamma)^2 + 4am} - (a-m-b\gamma)} - \frac{2am\gamma}{a + 2m + \frac{2m(b-a)}{m+a}\gamma}\right] \\
&= \frac{a(m-a)}{b(b-a)}\left(b - \frac{2am}{m-a}\right) > 0.
\end{aligned}$$

Then, as a consequence of Theorem 1.3, we obtain immediately

Corollary 1.5. *Suppose $d_1, d_2, a, m, b, \gamma$ are positive constants with $d_1 = d_2$, $m > a$ and $b > \frac{2am}{m-a}$. Then there exists a positive constant γ_0 depending only on b, a, m such that $\mathbf{E}_*$ is globally asymptotically stable for any $\gamma \geq \gamma_0$.*

The steady-states of system (1.1) satisfy

$$\begin{cases} -d_1\Delta u = au - u^2 - \frac{uv}{m+u}, & x \in \Omega, \\ -d_2\Delta v = bv - \frac{v^2}{\gamma u}, & x \in \Omega, \\ \partial_\nu u = \partial_\nu v = 0, & x \in \partial\Omega. \end{cases} \tag{1.3}$$

Theorems 1.1–1.3 obviously imply some conditions for the non-existence of positive non-constant solutions of system (1.3), which are independent of the coefficients d_1 and d_2. In [9], Peng and Wang gave some conditions for the non-existence of positive non-constant solutions of system (1.3), which depend on d_1 and d_2, see [9, Theorems 3.1 and 3.5]. For example, they proved that system (1.3) has no positive non-constant solution if d_1 and d_2 are sufficiently large, see [9, Theorems 3.1]. By using a different approach from those in literature (see e.g. [8, 10]), we prove the following result on the non-existence of positive non-constant solutions.

Theorem 1.6. *Suppose $m \geq a$. Then system (1.3) has no positive non-constant solution.*

We point out that the approach used to show Theorem 1.6 can be applied to some interesting models to discuss non-existence of positive non-constant solutions, for instance, the steady-state Sel'kov model (see [12]):

$$\begin{cases} -\theta\Delta u = \lambda(1 - uv^p), & x \in \Omega, \\ -\Delta v = \lambda(uv^p - v), & x \in \Omega, \\ \partial_\nu u = \partial_\nu v = 0, & x \in \partial\Omega, \end{cases} \tag{1.4}$$

where θ, λ, p are positive constants, which had been studied in [6, 8, 14]. For the case when $0 < p \leq 1$, Peng [8] proved the non-existence of positive non-constant solutions of system (1.3) if θ is sufficiently large. In the present paper, we remove the restriction on θ and obtain

Theorem 1.7. *Suppose θ, λ, p are positive constants. If $0 < p \leq 1$, then system (1.4) has no positive non-constant solution.*

The rest of this paper is organized as follows. In Section 2, we will prove Theorems 1.2 and 1.3 by using Lyapunov function method. In Section 3, we will prove Theorems 1.6 and 1.7 by a novel approach. Finally, our conclusions are given in Section 4.

2 Proofs of Theorems 1.2 and 1.3

We begin with the following lemma.

Lemma 2.1. *The following statements hold.*

(a) $m(m + u_*) > b\gamma u_*$ *if and only if* $m > M_1 = \frac{ab\gamma}{a+b\gamma}$.

(b) $\frac{am}{a+2m} < u_*$ *if and only if* $m > M_2 = \frac{1}{2}\big[(b\gamma - 2a)_+ + \sqrt{b\gamma(b\gamma - 2a)_+}\,\big]$, *where* $s_+ = \max\{0, s\}$.

Proof. As for the conclusion (a), it it clear to see that the case where $m \geq b\gamma$ is trivial. We now suppose $m < b\gamma$. For the case, if $m(m+u_*) > b\gamma u_*$, i.e., $m^2 > (b\gamma - m)u_*$, then

$$2m^2 - (b\gamma - m)(a - m - b\gamma) > (b\gamma - m)\sqrt{(a - m - b\gamma)^2 + 4am}, \tag{2.1}$$

and then taking the square on the two sides of (2.1) yields $m > \frac{ab\gamma}{a+b\gamma}$. Note that the above reasoning process is also inverse since $m > \frac{ab\gamma}{a+b\gamma}$ implies

$$\begin{aligned} 2m^2 - (b\gamma - m)(a - m - b\gamma) &= m^2 + (b\gamma)^2 + ma - ab\gamma \\ &> mb\gamma + ma - ab\gamma \\ &> 0. \end{aligned}$$

Thus the conclusion (a) is valid.

As for the conclusion (b), a simple calculation gives

$$\begin{aligned} (a + 2m)u_* - am > 0 \Leftrightarrow b\gamma &< \frac{2(a+m)^2}{a+2m} \\ &\Leftrightarrow 2m^2 + 2(2a - b\gamma)m + a(2a - b\gamma) > 0. \end{aligned}$$

Solving the latter gives $m > M_2$. This completes the proof of the lemma. □

Proof of Theorem 1.2. Let (u, v) be a positive solution of system (1.1). Adapting the Lyapunov function in [2,3], we define

$$\begin{aligned} V(u,v) &= \int_{u_*}^{u} \frac{\eta - u_*}{\eta g(\eta)} d\eta + \frac{\gamma u_*}{v_*} \int_{v_*}^{v} \frac{\eta - v_*}{\eta} d\eta, \text{ where } g(u) = \frac{u}{m+u}; \\ W(t) &= \int_\Omega V(u(x,t), v(x,t)) dx. \end{aligned} \tag{2.2}$$

Denote $g_1(u,v) = au - u^2 - g(u)v$ and $g_2(u,v) = bv - \frac{v^2}{\gamma u}$. Some calculations give

$$\begin{aligned} \int_\Omega V_u(u,v)u_t dx &= \int_\Omega \frac{u - u_*}{ug(u)} [d_1 \Delta u + g_1(u,v)] dx \\ &= -d_1 \int_\Omega [u_*(g(u) + ug'(u)) - u^2 g'(u)] \frac{|\nabla u|^2}{[ug(u)]^2} dx \\ &\quad + \int_\Omega \frac{u - u_*}{g(u)} \left[u_* - u + \left(\frac{g(u_*)}{u_*} - \frac{g(u)}{u} \right) v_* + \frac{g(u)}{u}(v_* - v) \right] dx \\ &= -d_1 \int_\Omega \frac{(u_* - m)u^2 + 2mu^* u}{(m+u)^2} \frac{|\nabla u|^2}{[ug(u)]^2} dx - \int_\Omega \frac{(u - u_*)(v - v_*)}{u} dx \\ &\quad - \int_\Omega \frac{(u - u_*)^2}{g(u)} \left[1 - \frac{b\gamma u_*}{(m+u)(m+u_*)} \right] dx \quad (\text{note that } v_* = b\gamma u_*), \end{aligned}$$

and

$$\begin{aligned} \int_\Omega V_v(u,v)v_t dx &= \frac{\gamma u_*}{v_*} \int_\Omega \frac{v - v_*}{v} [d_2 \Delta v + g_2(u,v)] dx \\ &= -\gamma d_2 u_* \int_\Omega \frac{|\nabla v|^2}{v^2} dx + \frac{u_*}{v_*} \int_\Omega (v - v_*) \left(\frac{v_*}{u_*} - \frac{v}{u} \right) dx \\ &= -\gamma d_2 u_* \int_\Omega \frac{|\nabla v|^2}{v^2} dx + \frac{u_*}{v_*} \int_\Omega (v - v_*) \left(\frac{v_*}{u_*} - \frac{v_*}{u} + \frac{v_*}{u} - \frac{v}{u} \right) dx \\ &= -\gamma d_2 u_* \int_\Omega \frac{|\nabla v|^2}{v^2} dx + \int_\Omega \frac{(v - v_*)(u - u_*)}{u} dx - \frac{u_*}{v_*} \int_\Omega \frac{(v - v_*)^2}{u} dx. \end{aligned}$$

It follows that

$$
\begin{aligned}
W'(t) = & -d_1 \int_\Omega \frac{(u_* - m)u^2 + 2mu_* u}{(m+u)^2} \frac{|\nabla u|^2}{[ug(u)]^2} dx - \gamma d_2 u_* \int_\Omega \frac{|\nabla v|^2}{v^2} dx \\
& - \int_\Omega \frac{(u-u_*)^2}{g(u)} \left[1 - \frac{b\gamma u_*}{(m+u)(m+u_*)}\right] dx - \frac{u_*}{v_*} \int_\Omega \frac{(v-v_*)^2}{u} dx \\
\leq & -d_1 \int_\Omega \frac{(u_* - m)u^2 + 2mu_* u}{(m+u)^2} \frac{|\nabla u|^2}{[ug(u)]^2} dx - \gamma d_2 u_* \int_\Omega \frac{|\nabla v|^2}{v^2} dx \\
& - \left[1 - \frac{b\gamma u_*}{m(m+u_*)}\right] \int_\Omega \frac{(u-u_*)^2}{g(u)} dx - \frac{u_*}{v_*} \int_\Omega \frac{(v-v_*)^2}{u} dx.
\end{aligned} \tag{2.3}
$$

We now assume that $m > \max\{M_1, M_2\}$. Then, $1 - \frac{b\gamma u_*}{m(m+u_*)} > 0$ by Lemma 2.1(a), and $\frac{am}{a+2m} < u_*$ by Lemma 2.1(b), so there exists a constant $\varepsilon > 0$ such that

$$
\frac{(a+\varepsilon)m}{a+\varepsilon+2m} < u_*. \tag{2.4}
$$

On the other hand, from (1.1), we have

$$
u_t - d_1 \Delta u \leq u(a-u), \quad \forall (x,t) \in \Omega \times (0, +\infty).
$$

It follows from the comparison principle that $\limsup_{t\to+\infty} \max_{x\in\overline{\Omega}} u(x,t) \leq a$, and hence there exists some $T > 0$, such that

$$
u(x,t) < a + \varepsilon, \quad \forall (x,t) \in \overline{\Omega} \times [T, +\infty). \tag{2.5}
$$

Combining (2.4) and (2.5) gives

$$
\begin{aligned}
(u_* - m)u^2(x,t) + 2mu_* u(x,t) &= u(x,t)[u(x,t) + 2m] \left[u_* - \frac{mu(x,t)}{u(x,t) + 2m}\right] \\
&> u(x,t)[u(x,t) + 2m] \left[u_* - \frac{m(a+\varepsilon)}{a+\varepsilon+2m}\right] \\
&> 0, \quad \forall (x,t) \in \overline{\Omega} \times [T, +\infty),
\end{aligned}
$$

therefore, $W'(t) \leq 0$ for all $t \geq T$, and equality holds if and only if $(u,v) = \mathbf{E}_*$, so $\mathbf{E}_*$ is globally attractive. Since $m > M_1$ (i.e., $m(a+b\gamma) > ab\gamma$), $\mathbf{E}_*$ is locally asymptotically stable according to Theorem 1.1(a), so is globally asymptotically stable. The proof of the theorem is complete. □

We now are ready to show Theorem 1.3, whose proof is based on the following lemma.

Lemma 2.2. *Suppose $d_1 = d_2$ and $b > a$. Then*

$$
\limsup_{t\to+\infty} \max_{x\in\overline{\Omega}} u(x,t) \leq a\left(1 + \frac{b-a}{m+a}\gamma\right)^{-1}.
$$

Proof. Like in [11], we set $\varphi = \frac{v}{u}$. Then a simple calculation gives

$$
\varphi_t = \frac{1}{u} v_t - \frac{v}{u^2} u_t, \qquad \nabla \varphi = \frac{1}{u} \nabla v - \frac{v}{u^2} \nabla u,
$$

and

$$\Delta\varphi = \frac{1}{u}\Delta v - \frac{v}{u^2}\Delta u - \frac{2}{u}\nabla u \cdot \nabla\varphi,$$

so that

$$\begin{aligned}\varphi_t - \frac{2d_1}{u}\nabla u \cdot \nabla\varphi - d_1\Delta\varphi &= \varphi\left(b - a - \frac{1}{\gamma}\varphi + u + \frac{v}{m+u}\right)\\ &\geq \varphi\left(b - a - \frac{1}{\gamma}\varphi\right),\end{aligned}$$

therefore, from the comparison principle, for any $0 < \varepsilon \ll 1$ there exists some constant $T_1^\varepsilon \gg 1$ such that

$$\varphi(x,t) \geq (b-a)\gamma - \varepsilon > 0, \quad \forall (x,t) \in \overline{\Omega} \times [T_1^\varepsilon, +\infty). \tag{2.6}$$

By a similar argument to (2.5), there exists some constant $T_2^\varepsilon > T_1^\varepsilon$ such that

$$u(x,t) < a + \varepsilon, \quad \forall (x,t) \in \overline{\Omega} \times [T_2^\varepsilon, +\infty). \tag{2.7}$$

Combining (2.6), (2.7) and $(1.1)_1$, we obtain

$$u_t - d_1\Delta u \leq u\left[a - \left(1 + \frac{(b-a)\gamma - \varepsilon}{m + a + \varepsilon}\right)u\right], \quad \forall (x,t) \in \overline{\Omega} \times [T_2^\varepsilon, +\infty).$$

This implies $\limsup_{t\to+\infty} \max_{x\in\overline{\Omega}} u(x,t) \leq \frac{a}{1+\frac{(b-a)\gamma-\varepsilon}{m+a+\varepsilon}}$. Then, letting $\varepsilon \to 0$ gives the desired result. □

Proof of Theorem 1.3. We adapt the same Lyapunov function as that in (2.2).

From Lemma 2.2 and (1.2), there exist some constants $0 < \varepsilon \ll 1$ and $T \gg 1$ such that

$$u(x,t) \leq a\left[1 + \frac{(b-a)\gamma - \varepsilon}{m+a}\right]^{-1}, \quad \forall (x,t) \in \overline{\Omega} \times [T, \infty), \tag{2.8}$$

and

$$\begin{aligned}2am\left\{a + 2m + \frac{2m[(b-a)\gamma - \varepsilon]}{m+a}\right\}^{-1} &< a - m - b\gamma + \sqrt{(a - m - b\gamma)^2 + 4am}\\ &= 2u_*.\end{aligned} \tag{2.9}$$

Since $F(x) = x/(2m + x)$ is increasing in $[0, \infty)$, it follows from (2.8) and (2.9) that

$$\frac{mu(x,t)}{2m + u(x,t)} \leq \frac{am}{a + 2m + \frac{2m[(b-a)\gamma-\varepsilon]}{m+a}} < u_*, \quad \forall (x,t) \in \overline{\Omega} \times [T, +\infty).$$

That is,

$$(u_* - m)u^2 + 2muu_* = u\left[u_*(u + 2m) - mu\right] > 0, \quad \forall (x,t) \in \overline{\Omega} \times [T, +\infty).$$

Combining this and (2.3) with $d_1 = d_2$ yields $W'(t) \leq 0$ for all $t \geq T$, and equality holds if and only if $(u, v) = \mathbf{E}_*$, so $\mathbf{E}_*$ is globally attractive. Since $m > M_1$, $\mathbf{E}_*$ is locally asymptotically stable according to Theorem 1.1(a), so is globally asymptotically stable. The proof is complete. □

3 Proofs of Theorems 1.6 and 1.7

We first show Theorem 1.6.

Proof of Theorem 1.6. Assume that (u, v) is a positive solution of system (1.3). Multiplying $(1.1)_1$ by $[(a-u)(m+u)-v]$ and integrating by parts over Ω, we have

$$d_1 \int_\Omega \nabla u \cdot \nabla[(a-u)(m+u)-v]dx = \int_\Omega \frac{u}{m+u}[(a-u)(m+u)-v]^2 dx,$$

that is,

$$d_1 \int_\Omega (a-m-2u)|\nabla u|^2 dx - d_1 \int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega \frac{u}{m+u}[(a-u)(m+u)-v]^2 dx. \tag{3.1}$$

Multiplying $(1.1)_2$ by $(u - \frac{v}{b\gamma})$ and integrating over Ω, we obtain

$$d_2 \int_\Omega \nabla u \cdot \nabla v dx - \frac{d_2}{b\gamma} \int_\Omega |\nabla v|^2 dx = \int_\Omega \frac{bv}{u}\left(u - \frac{v}{b\gamma}\right)^2 dx. \tag{3.2}$$

We first multiply (3.2) by d_1/d_2, and then add the resulting equation and (3.1) to get

$$\begin{aligned} &d_1 \int_\Omega (a-m-2u)|\nabla u|^2 dx - \frac{d_1}{b\gamma} \int_\Omega |\nabla v|^2 dx \\ &\quad - \int_\Omega \left\{ \frac{u}{m+u}[(a-u)(m+u)-v]^2 + \frac{d_1 bv}{d_2 u}\left(u - \frac{v}{b\gamma}\right)^2 \right\} dx = 0. \end{aligned} \tag{3.3}$$

Since $m \geq a$, the first term on the left hand side of (3.3) is non–positive and hence u and v must be constants. The proof is complete. □

Proof of Theorem 1.7. Assume that (u, v) is a positive solution of system (1.4). Multiplying (1.4) by $(\frac{1}{u} - v^p)$ and $(uv^{p-1} - 1)$, respectively, and integrating by parts over Ω, we have

$$-\theta \int_\Omega \frac{|\nabla u|^2}{u^2} dx - \theta \int_\Omega \nabla u \cdot \nabla v^p dx = \lambda \int_\Omega u\left(\frac{1}{u} - v^p\right)^2 dx, \tag{3.4}$$

and

$$(p-1) \int_\Omega uv^{p-2}|\nabla v|^2 dx + \frac{1}{p} \int_\Omega \nabla u \cdot \nabla v^p dx = \lambda \int_\Omega v(uv^{p-1}-1)^2 dx. \tag{3.5}$$

We first multiply (3.5) by $p\theta$, and then add the resulting equation and (3.4) to obtain

$$\int_\Omega \left[\theta \frac{|\nabla u|^2}{u^2} + \theta p(1-p)uv^{p-2}|\nabla v|^2\right] dx + \lambda \int_\Omega \left[u\left(\frac{1}{u} - v^p\right)^2 + p\theta v(uv^{p-1}-1)^2\right] dx = 0.$$

Consequently, u and v must be constants if $p \in (0, 1]$. The proof is complete. □

Remark 3.1. In [14, Remark 2.1], the authors pointed out that it is difficult to expect the bifurcation of (1.4) near $(u, v) = (1, 1)$ if $0 < p \leq 1$ since the constant positive solution $(u, v) = (1, 1)$ is uniformly asymptotically stable for the corresponding reaction–diffusion system to (1.4) for the case. Our Theorem 1.7 shows that no bifurcation will happen for system (1.4) provided that $0 < p \leq 1$.

4 Conclusions

In this paper, we prove some new global stability results. In particular, the works by Chen and Shi [1] and Duan, Niu and Wei [2], mentioned above, have been improved. In addition, we derive a non-existence result of the positive non-constant steady-states for system (1.1) by using a different approach from those in literature. By virtue of the approach, we also obtain a complete understanding of the steady-state Sel'kov model for the case when $0 < p \leq 1$.

Acknowledgements

We would like to express deep thanks to the referee for their careful reading of the manuscript and important comments which greatly improve the paper. The research was supported in part by the NSFC (grants 12071058, 11971496) and the Program for Liaoning Innovative Talents in University (grant LR2016004).

Appendix

In this part, we will only prove the global existence of positive solutions of problem (1.1) since the proof to uniqueness is standard. To this end, we will use the regularization method. In what follows, we assume that the initial data u_0 and v_0 are $C^1(\overline{\Omega})$ functions satisfying $u_0, v_0 > 0$ on $\overline{\Omega}$ and $\partial_\nu u_0 = \partial_\nu v_0 = 0$ on $\partial\Omega$.

Let $\varepsilon \in (0,1)$ be a constant. Consider the regularized problem:

$$\begin{cases} u_t - d_1\Delta u = au - u^2 - \dfrac{uv}{m+u}, & x \in \Omega, t > 0, \\ v_t - d_2\Delta v = bv - \dfrac{v^2}{\gamma(u+\varepsilon)}, & x \in \Omega, t > 0, \\ \partial_\nu u = \partial_\nu v = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \overline{\Omega}. \end{cases} \tag{$(P)_\varepsilon$}$$

From the standard theory of parabolic equations, system $(P)_\varepsilon$ has a unique nonnegative global solution $(u_\varepsilon, v_\varepsilon)$ for any given $\varepsilon \in (0,1)$.

Let $\overline{u}(t)$ be a solution of the following problem:

$$\begin{cases} \dfrac{d\overline{u}}{dt} = a\overline{u} - \overline{u}^2, \quad t > 0, \\ \overline{u}(0) = \max\limits_{\overline{\Omega}} u_0(x) =: M > 0. \end{cases}$$

It is easy to check that $\overline{u}(t) = \frac{e^{at}}{M^{-1} + \int_0^t e^{as}ds} \leq M_1$ on $[0, +\infty)$ for some constant M_1 independent of ε. Note that u_ε satisfies

$$(u_\varepsilon)_t - d_1\Delta u_\varepsilon \leq u_\varepsilon(a - u_\varepsilon), \qquad x \in \Omega, t > 0.$$

It follows from the comparison principle that $u_\varepsilon(x,t) \leq \overline{u}(t) \leq M_1$ on $\overline{\Omega} \times [0, +\infty)$. Consequently, we have

$$(v_\varepsilon)_t - d_2\Delta v_\varepsilon \leq bv_\varepsilon - \frac{v_\varepsilon^2}{\gamma(M_1+1)}, \qquad x \in \Omega,\ t > 0.$$

Similarly, there exists some constant $M_2 > 0$, independent of ε, such that $v_\varepsilon(x,t) \le M_2$ on $\overline{\Omega} \times [0,+\infty)$. Hence,

$$(u_\varepsilon)_t - d_1\Delta u_\varepsilon \ge -(M_1 + M_2 m^{-1})u_\varepsilon =: -C_1 u_\varepsilon, \qquad x \in \Omega, t > 0.$$

By the comparison principle, $u_\varepsilon(x,t) \ge \underline{u}(t)$ on $\overline{\Omega} \times [0,+\infty)$, where $\underline{u}(t) = (\min_{\overline{\Omega}} u_0)e^{-C_1 t}$ satisfies

$$\begin{cases} \frac{d\underline{u}}{dt} = -C_1\underline{u}, & t > 0, \\ \underline{u}(0) = \min_{\overline{\Omega}} u_0 > 0. \end{cases}$$

It follows that

$$(v_\varepsilon)_t - d_2\Delta v_\varepsilon \ge -\frac{M_2 e^{C_1 t}}{\gamma \min_{\overline{\Omega}} u_0} v_\varepsilon =: -C_2 e^{C_1 t} v_\varepsilon, \qquad x \in \Omega,\ t > 0.$$

Again using the comparison principle, we see that $v_\varepsilon(x,t) \ge \underline{v}(t)$ on $\overline{\Omega} \times [0,+\infty)$, where $\underline{v}(t) = (\min_{\overline{\Omega}} v_0)e^{-C_2 \int_0^t e^{C_1 s} ds}$ satisfies

$$\begin{cases} \frac{d\underline{v}}{dt} = -C_2 e^{C_1 t}\underline{v}, & t > 0, \\ \underline{v}(0) = \min_{\overline{\Omega}} v_0 > 0. \end{cases}$$

In summary, we have, for all $\varepsilon \in (0,1)$,

$$M_1 \ge u_\varepsilon(x,t) \ge \underline{u}(t), \qquad M_2 \ge v_\varepsilon(x,t) \ge \underline{v}(t), \qquad \forall (x,t) \in \overline{\Omega} \times [0,+\infty).$$

Then, by a standard compactness argument, one can obtain a positive global solution of system (1.1). This proof is complete.

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