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On the existence of ground state solutions for the logarithmic Schrödinger–Bopp–Podolsky system

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Abstract. This paper deals with the following logarithmic Schrödinger–Bopp–Podolsky system

$$\begin{cases} -\Delta u + V(x)u - \phi u = u \log u^2 & \text{in } \mathbb{R}^3, \\ -\Delta \phi + \Delta^2 \phi = 4\pi u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $V(x) \in C(\mathbb{R}^3, \mathbb{R})$. By using the variational method developed by Szulkin for the functional which is the sum of a smooth and a convex lower semicontinuous term, we study the properties of the solutions for the above system under different potential conditions. When the potential is coercive, we discuss the existence of a ground state solution. Moreover, we also consider the cases where $V(x)$ is periodic or asymptotically periodic, and obtain a ground state solution in each scenario, respectively.

Keywords: Schrödinger–Bopp–Podolsky system, logarithmic nonlinearity, variational methods.

2020 Mathematics Subject Classification: 35J20, 35J50, 35J40.

1 Introduction and main results

In the past few decades, the following nonlinear Schrödinger–Bopp–Podolsky system

$$\begin{cases} -\Delta u + V(x)u + \lambda \phi u = f(u) & \text{in } \mathbb{R}^3, \\ -\Delta \phi + a^2 \Delta^2 \phi = 4\pi u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where λ, $a \in \mathbb{R}$ are parameters, has been studied by many researchers. This system is closely related to the Bopp–Podolsky electromagnetic theory and arises when coupling a Schrödinger field $\psi = \psi(t, x)$ with its electromagnetic field in the Bopp–Podolsky electromagnetic theory. Among them, the Bopp–Podolsky theory is called a second-order gauge theory of electromagnetic fields, which was established by Bopp [3] and later developed by Podolsky [22] to solve

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the so-called infinity problem in the classical Maxwell theory. From the viewpoint of electromagnetic fields, the Bopp–Podolsky theory can be explained as an effective theory for short distances, while for large distances it is experimentally indistinguishable from the Maxwell theory (see [9]).

To our knowledge, d'Avenia and Siciliano in [7] first proved the existence and non-existence of nontrivial solutions for system (1.1) with constant potential via using the variational method and splitting lemma, where the nonlinear term is denoted as $f(u) = |u|^{p-2}u$. Moreover, in the radial case, they found that solutions tend to those of the classical Schrödinger–Poisson system as $a \to 0$. In [13], Li, Pucci and Tang generalized the existing results for system (1.1) to the critical case and used the Pohožaev–Nehari manifold method to divide the equation into two cases, the constant potential and the asymptotic constant potential, proving that there is a ground state solution when the nonlinear term increases critically. Then in [4], the authors improved the nonlinear term in [13] to a general nonlinear term, and adopted some new analytical techniques and new inequalities to prove the existence of solutions in different cases. Yang, Chen and Liu [27] applied a cut-off function, the mountain pass theorem and Moser iteration to prove the existence of nontrivial solution for system (1.1) with critical growth. Zhu, Chen and Chen [30] studied the existence of different solutions for system (1.1) under nonlinearity effects.

Later, Jia, Li and Chen [12] established the existence of ground state solutions for nonautonomous Schrödinger–Bopp–Podolsky system. Liu and Chen [18] studied the existence, nonexistence and asymptotic behavior of ground state solutions for problem (1.1) with critical Sobolev exponent. Peng [20] proved the existence and multiplicity of solutions for the system (1.1). Zhang [28] investigated the existence of sign-changing solutions for system (1.1) with general nonlinearity. Yang, Yuan and Liu [26] were concerned with the existence of ground states for a nonlinear Schrödinger–Bopp–Podolsky system with asymptotically periodic potentials. Li and Zhang [14] found the existence of normalized solution for the system (1.1). For more information on the results of a system like (1.1), the readers can refer to [1,5,10,19,29] and the references therein.

Recently, the Schrödinger problem with logarithmic terms given by

$$i\varepsilon \frac{\partial \Phi}{\partial t} = -\varepsilon^2 \Delta \Phi + W(x)\Phi - \Phi \log |\Phi|^2, \quad N \geq 3, \tag{1.2}$$

where $\Phi\colon [0, +\infty) \times \mathbb{R}^N \to \mathbb{C}$, has also received extensive attention due to its physical influence, such as quantum mechanics, quantum optics, nuclear physics, effective quantum, and Bose–Einstein condensation (see [31]). The standing wave solutions for problem (1.2) have the ansatz form $\Phi(t, x) = u(x)\mathrm{e}^{-i\omega t/\varepsilon}$, which leads to the following equation

$$-\varepsilon^2 \Delta u + V(x)u = u \log u^2, \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

where $V(x) = W(x) - \omega$ and $\omega \in \mathbb{R}$. Its associated energy functional is as follows

$$\tilde{\mathcal{J}}_\varepsilon(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left(\varepsilon^2 |\nabla u|^2 + (V(x) + 1)u^2\right) \mathrm{d}x - \frac{1}{2} \int_{\mathbb{R}^N} u^2 \log u^2 \, \mathrm{d}x. \tag{1.4}$$

From a mathematical point of view, problem like (1.3) is very interesting because it is not like the general Schrödinger equation whose energy functional is C^1 class, while the energy functional $\tilde{\mathcal{J}}_\varepsilon(u)$ of the logarithmic Schrödinger equation is non-smooth. Therefore, it cannot be solved by the general critical point theory, and new technical means must be employed for research, which brings many difficulties to the research process. In fact, there have been some results in this direction.

In [6], d'Avenia, Montefusco and Squassina used the non-smooth critical point theory to demonstrate the existence of infinitely many weak solutions for the logarithmic Schrödinger equation (1.3) as $\varepsilon = 1$ and $V(x) = 1$. Furthermore, they have proven that there exists a unique positive solution that is radially symmetric and nondegenerate. In [8], d'Avenia, Squassina and Zenari adopted the same way to show the existence of infinitely many solutions of a fractional Schrödinger equation with logarithmic nonlinearity.

Squassina and Szulkin [23] combined the method of the minimax principles for lower semicontinuous functionals proposed by Szulkin [24] to study the following problem

$$-\Delta u + V(x)u = Q(x)u \log u^2, \quad \text{in } \mathbb{R}^N, \tag{1.5}$$

where $V, Q\colon \mathbb{R}^N \to \mathbb{R}$ are 1-periodic continuous functions for x verifying

$$\min_{x\in\mathbb{R}^N} Q(x) > 0 \quad \text{and} \quad \min_{x\in\mathbb{R}^N} (V+Q)(x) > 0.$$

As a consequence, they obtained the existence of infinitely many geometrically distinct solutions for problem (1.5). In [11], Ji and Szulkin, inspired by [23], discussed the existence of multiple solutions and a ground state solution for equation (1.3), where $\varepsilon = 1$ and the potential $V(x)$ satisfies

$$V(x) \in C\left(\mathbb{R}^N, \mathbb{R}\right), \quad \lim_{|x|\to+\infty} V(x) = V_\infty \quad \text{and} \quad V_\infty + 1 \in (0, +\infty].$$

In the recent paper [2], Alves and de Morais Filho investigated the existence of positive solutions for the logarithmic elliptic equation (1.3). By using the variational method developed by Szulkin, they got the existence and concentration of solutions as $\varepsilon \to 0$, and they also considered the cases when the potential $V(x)$ is periodic or asymptotically periodic.

In particular, Peng and Jia [21] studied the logarithmic Schrödinger–Bopp–Podolsky system as follows

$$\begin{cases} -\varepsilon^2\Delta u + V(x)u - \phi u = u \log u^2 & \text{in } \mathbb{R}^3, \\ -\varepsilon^2\Delta\phi + \varepsilon^4\Delta^2\phi = 4\pi u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.6}$$

where V satisfies the following global condition:

$$(V)V(x) \in C\left(\mathbb{R}^3, \mathbb{R}\right) \quad \text{and} \quad V_\infty := \lim_{|x|\to+\infty} V(x) > \inf_{x\in\mathbb{R}^3} V(x) = V_0 > -1.$$

Borrowing an idea from [2], they proved the existence and concentration behavior of positive solution for equation (1.6).

Inspired by the above literatures, in this paper, we establish the existence of ground state solutions for the following logarithmic Schrödinger–Bopp–Podolsky system under different potentials

$$\begin{cases} -\Delta u + V(x)u - \phi u = u \log u^2 & \text{in } \mathbb{R}^3, \\ -\Delta\phi + \Delta^2\phi = 4\pi u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.7}$$

where $V(x) \in C\left(\mathbb{R}^3, \mathbb{R}\right)$ satisfies the following global conditions:

(V_1) $V(x)$ is coercive and $-1 < V_0 = \inf_{x\in\mathbb{R}^3} V(x)$.

(V_2) $V(x)$ is a continuous $\mathbb{Z}^3$-periodic function, i.e.,

$$V(x+y)=V(x),\quad \forall x\in\mathbb{R}^3,\forall y\in\mathbb{Z}^3 \quad \text{and} \quad -1<V_0=\inf_{x\in\mathbb{R}^3}V(x).$$

(V_3) $V(x)$ is a continuous asymptotically periodic, that is, there is a continuous $\mathbb{Z}^3$-periodic function V_p such that

$$-1<V_0=\inf_{x\in\mathbb{R}^3}V(x)\le V(x)<V_p(x),\quad \text{for all } x\in\mathbb{R}^3$$

and

$$\left|V(x)-V_p(x)\right|\to 0 \quad \text{as } |x|\to+\infty.$$

We will use the Variational Method developed by Szulkin [24] to get our results. Compared with previous papers, the problem we study is more complex due to the interaction between logarithmic and non-local terms. To our knowledge, there is currently only one article [21] in this direction, but in that literature the authors considered the case of asymptotic constant potential, which is different from the problem we are studying. Therefore, it is necessary to employ different methods to address our research question.

Definition 1.1. A weak solution for system (1.7) means a pair of $(u,\phi)\in E_V\times\mathcal{D}$ satisfying $u^2\log u^2\in L^1\left(\mathbb{R}^3\right)$ (i.e., $I(u)<+\infty$) and

$$\int_{\mathbb{R}^3}(\nabla u\nabla\zeta+V(x)u\zeta-\phi u\zeta)\mathrm{d}x=\int_{\mathbb{R}^3}u\zeta\log u^2\,\mathrm{d}x,\quad \text{for any } \zeta\in C_0^\infty\left(\mathbb{R}^3\right),$$

$$\int_{\mathbb{R}^3}\nabla\phi\nabla\xi\mathrm{d}x+\int_{\mathbb{R}^3}\Delta\phi\Delta\xi\mathrm{d}x=4\pi\int_{\mathbb{R}^3}\xi u^2\,\mathrm{d}x,\quad \text{for any } \xi\in\mathcal{D},$$

where the definitions of $I(u)$, E_V and $\mathcal{D}$ will be given in Sect. 2.

We derive the following results.

Theorem 1.2. *Assume that the condition (V_1) holds. Then, system (1.7) has one ground state solution.*

Theorem 1.3. *Assume that the condition (V_2) holds. Then, system (1.7) has one ground state solution.*

Theorem 1.4. *Assume that the condition (V_3) holds. Then, system (1.7) has one ground state solution.*

Remark 1.5. In this article, we just consider the case that $V_p<+\infty$ in the condition (V_3). Indeed, the case $V_p=+\infty$ is simpler since the embedding $E_V\hookrightarrow L^s(\mathbb{R}^3)$ is compact for $s\in[2,6)$, where E_V see Sect. 2. Moreover, we should note that $V_0>-1$ is assumed instead of the general condition $V_0>0$ because the nonlinear term we consider is logarithmic. In fact, this question can be explained from (1.4), and readers can also refer to [25] for more details.

Remark 1.6. When considering the logarithmic equation, we will cite the useful logarithmic Sobolev inequality found in [15]:

$$\int_{\mathbb{R}^3}u^2\log u^2\,\mathrm{d}x\le\frac{a^2}{\pi}\|\nabla u\|_2^2+\left(\log\|u\|_2^2-3(1+\log a)\right)\|u\|_2^2,\quad \text{for all } a>0. \tag{1.8}$$

Let us give a sketch of the proof of the results and explain the difficulties encountered in the process of solving them.

- Due to the existence of the logarithmic term, the energy functional $I(u)$ [see (2.3)] related to system (1.7) may take the value $+\infty$ since there is a function $u \in H^1(\mathbb{R}^3)$ such that $\int_{\mathbb{R}^3} u^2 \log u^2 \, dx = -\infty$. Therefore, the functional $I(u)$ is not of class C^1, which makes it impossible to be solved by the general critical point theory.

- In order to find solutions of system (1.7), similar to [23], we will use a technical decomposition of $I(u)$ [see (2.4)]. In this case, we can apply the Mountain Pass Theorem without (PS) condition for a functional that is the sum of a smooth and a convex lower semicontinuous term, which was first mentioned in [2].

- Because of the interaction between the logarithmic term and the non-local term, the boundedness of (PS) sequences is more difficult to obtain than in [2], so we solve this problem with the help of literature [21].

- For the proof of Theorem 1.2, since the compactness can be directly obtained under the coercive potential condition, we only need to prove the boundness of (PS) sequence and combine the Mountain Pass Theorem without (PS) condition to complete the proof.

- The lack of compactness makes the proofs of Theorems 1.3 and 1.4 more difficult. Under the periodic potential condition, we still begin by decomposing the functional and defining the Nehari manifold $\mathcal{M}_V$ [see (3.6)]. Subsequently, we derive the required conclusions via combining the Mountain Pass Theorem without (PS) condition. During this process, it is necessary to refer to [17] to return to the solution of the original problem [see Lemma 5.1]. The key steps in the proof involve verifying that the mountain pass value is equal to the infimum on the Nehari manifold, i.e., $c = c_V := \inf_{u \in \mathcal{M}_V} I(u)$ [see Lemma 3.7], and $u_n \rightharpoonup u_0 \neq 0$. However, the difference of asymptotic periodic potential is that the proof of $u_n \rightharpoonup u_0 \neq 0$ is obtained via comparing c_V with c_{V_p} [see Lemma 6.2], and other proofs are similar to the periodic potential case.

This paper is organized as follows. In Section 2, we recall some lemmas which we will use in the paper. In Section 3, we introduce the Mountain Pass Theorem without (PS) condition. In Section 4, we give the proof of Theorem 1.2. Subsequently, Theorems 1.3 and 1.4 will be proved in Sections 5 and 6, respectively.

Throughout this paper, we use the following notations:

- $L^s(\mathbb{R}^3)$ $(2 \leq s < \infty)$ denotes the Lebesgue space with the norm $\|u\|_s = \left(\int_{\mathbb{R}^3} |u|^s \, dx\right)^{1/s}$;

- $B_r(x)$ denotes the ball centered at x with radius r;

- $C, C', \tilde{C}, C_i$ $(i = 1, 2, \dots)$ denote positive constants possibly different in different places;

- $o_n(1)$ denotes a real sequence with $o_n(1) \to 0$ as $n \to +\infty$;

- $H_c^1(\mathbb{R}^3) = \{u \in H^1(\mathbb{R}^3) \mid u \text{ has a compact support}\}$.

2 Preliminaries

In this section, we present the variational setting and give a special decomposition of the functional $I(u)$, which needs to be adjusted since $I(u)$ may not be well defined in $H^1(\mathbb{R}^3)$. According to the research techniques in [23], we decompose $I(u)$ into a sum of a C^1 functional plus a convex lower semicontinuous functional.

2.1 Variational setting

Let $H^1(\mathbb{R}^3)$ denote the Sobolev space equipped with the norm

$$\|u\|_H = \left(\int_{\mathbb{R}^3} \left(|\nabla u|^2 + u^2 \right) \mathrm{d}x \right)^{\frac{1}{2}},$$

and $\mathcal{D}^{1,2}(\mathbb{R}^3) = \left\{ u \mid u \in L^6(\mathbb{R}^3), \nabla u \in L^2(\mathbb{R}^3) \right\}$ be endowed with the norm

$$\|u\|_{\mathcal{D}^{1,2}} = \left(\int_{\mathbb{R}^3} |\nabla u|^2 \, \mathrm{d}x \right)^{\frac{1}{2}}.$$

Let $\mathcal{D}$ be the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|\cdot\|_{\mathcal{D}}$ induced by the scalar product

$$\langle \phi, \xi \rangle_{\mathcal{D}} := \int_{\mathbb{R}^3} \nabla\phi \nabla\xi \mathrm{d}x + \int_{\mathbb{R}^3} \Delta\phi \Delta\xi \mathrm{d}x.$$

Clearly, $\mathcal{D}$ is a Hilbert space continuously embedded into $\mathcal{D}^{1,2}(\mathbb{R}^3)$ and consequently in $L^6(\mathbb{R}^3)$. Moreover, the space $\mathcal{D}$ is continuously embedded in $L^\infty(\mathbb{R}^3)$ (see Lemma 3.1 in [7]).

Next, we present the basic lemma.

Lemma 2.1 ([7, Lemma 3.2]). *The space $C_0^\infty(\mathbb{R}^3)$ is dense in*

$$\mathcal{A} := \left\{ \phi \in \mathcal{D}^{1,2}(\mathbb{R}^3) : \Delta\phi \in L^2(\mathbb{R}^3) \right\}$$

normed by $\sqrt{\langle \phi, \phi \rangle_{\mathcal{D}}}$ and, therefore, $\mathcal{D} = \mathcal{A}$.

In view of the Riesz Theorem, for every fixed $u \in H^1(\mathbb{R}^3)$, there exists a unique solution $\phi_u \in \mathcal{D}$ of the second equation in (1.7), namely,

$$-\Delta\phi + \Delta^2\phi = 4\pi u^2$$

and ϕ_u can be represented by

$$\phi_u = \int_{\mathbb{R}^3} \frac{1 - \mathrm{e}^{-|x-y|}}{|x-y|} u^2(y) \mathrm{d}y. \tag{2.1}$$

Then we have the following fundamental properties.

Lemma 2.2 ([7, Lemma 3.4] and [27, Lemma 2.2]). *For $u \in H^1(\mathbb{R}^3) \setminus \{0\}$, we have:*

(1) $\phi_u \geq 0$ in $\mathbb{R}^3$;

(2) $\phi_u \in L^s(\mathbb{R}^3) \cap C_0(\mathbb{R}^3)$, $\forall s \in (3, +\infty]$;

(3) $\phi_{u(\cdot+y)} = \phi_u(\cdot + y)$, $\forall y \in \mathbb{R}^3$;

(4) $\|\phi_u\|_{\mathcal{D}} \leq C\|u\|_H^2$, $\int_{\mathbb{R}^3} \phi_u u^2 \, \mathrm{d}x \leq C\|u\|_{12/5}^4$;

(5) if $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, then $\phi_{u_n} \rightharpoonup \phi_u$ in $\mathcal{D}$;

(6) if $u_n \to u$ in $L^{\frac{12}{5}}(\mathbb{R}^3)$, then $\phi_{u_n} \to \phi_u$ in $\mathcal{D}$ and $\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \, \mathrm{d}x \to \int_{\mathbb{R}^3} \phi_u u^2 \, \mathrm{d}x$;

(7) ϕ_u *is the unique minimizer of the functional*

$$X(\phi) = \frac{1}{2}\|\nabla\phi\|_2^2 + \frac{1}{2}\|\Delta\phi\|_2^2 - 4\pi\int_{\mathbb{R}^3}\phi u^2\,\mathrm{d}x, \quad \phi\in\mathcal{D}.$$

Now we introduce the working space,

$$E_V := \left\{u\in H^1\left(\mathbb{R}^3\right) : \int_{\mathbb{R}^3} V(x)u^2\,\mathrm{d}x < +\infty\right\}$$

with the norm $\|u\|_V$ induced by the scalar product

$$\langle u,v\rangle_V = \int_{\mathbb{R}^3}(\nabla u\nabla v + (V(x)+1)uv)\,\mathrm{d}x.$$

Note that E_V is a Hilbert space, and the embedding $E_V \hookrightarrow H^1\left(\mathbb{R}^3\right)$ is continuous.

For any $(u,\phi)\in E_V\times\mathcal{D}$, the associated energy functional of system (1.7) is given by

$$\begin{aligned}\mathcal{S}(u,\phi) = \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2 + (V(x)+1)u^2\right)\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^3}\phi u^2\,\mathrm{d}x + \frac{1}{16\pi}\int_{\mathbb{R}^3}|\nabla\phi|^2\,\mathrm{d}x\\ + \frac{1}{16\pi}\int_{\mathbb{R}^3}|\Delta\phi|^2\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^3}u^2\log u^2\,\mathrm{d}x.\end{aligned} \tag{2.2}$$

To avoid the difficulties caused by the strong uncertainty of the functional, we use the usual reduction function procedure introduced in [7]. From this, we can obtain that the reduced functional has the form as follows

$$\begin{aligned}I(u) := \mathcal{S}(u,\phi_u) = \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2 + (V(x)+1)u^2\right)\mathrm{d}x\\ - \frac{1}{4}\int_{\mathbb{R}^3}\phi_u u^2\,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^3}u^2\log u^2\,\mathrm{d}x.\end{aligned} \tag{2.3}$$

Remark 2.3. The following conclusions are equivalent:

(i) the pair $(u,\phi)\in E_V\times\mathcal{D}$ is a critical point of $\mathcal{S}$, namely, (u,ϕ) is a solution of (1.7);

(ii) u is a critical point of $I(u)$ and $\phi = \phi(u)$.

2.2 Decomposition of the functional $I(u)$

For $\delta > 0$ small, let us define the following functions:

$$F(s) = \begin{cases} 0, & s = 0,\\ -\frac{1}{2}s^2\log s^2, & 0<|s|<\delta,\\ -\frac{1}{2}s^2\left(\log\delta^2 + 3\right) + 2\delta|s| - \frac{1}{2}\delta^2, & |s|\geq\delta\end{cases}$$

and

$$G(s) = \begin{cases} 0, & |s|<\delta,\\ \frac{1}{2}s^2\log\left(s^2/\delta^2\right) + 2\delta|s| - \frac{3}{2}s^2 - \frac{1}{2}\delta^2, & |s|\geq\delta.\end{cases}$$

Then,

$$G(s) - F(s) = \frac{1}{2}s^2\log s^2, \quad \forall s\in\mathbb{R}$$

and the functional $I : E_V \to (-\infty, +\infty]$ may be denoted as

$$I(u) = \Phi(u) + \Psi(u), \quad u \in E_V, \tag{2.4}$$

where

$$\Phi(u) := \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + (V(x)+1)u^2\right)\, \mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2\, \mathrm{d}x - \int_{\mathbb{R}^3} G(u)\, \mathrm{d}x$$

and

$$\Psi(u) := \int_{\mathbb{R}^3} F(u)\, \mathrm{d}x.$$

As proven in [11], we can list some properties of F and G as follows:

- $F, G \in C^1(\mathbb{R}, \mathbb{R})$.
- If $\delta > 0$ is fixed and small enough, F is a nonnegative, convex, even function and $F'(s)s \geq 0$, for all $s \in \mathbb{R}$.
- For every fixed $p \in (2, 6)$, there exists $C > 0$ such that for any $s \in \mathbb{R}$,

$$\left|G'(s)\right| \leq C\,|s|^{p-1}. \tag{2.5}$$

Therefore, referring to [11], we can get that Ψ is nonnegative, convex and lower semicontinuous, and $\Phi \in C^1(E_V, \mathbb{R})$. Next, we will review some definitions and results of convex analysis that first appeared in [24].

Definition 2.4. Let E be a Banach space, E' be the dual space of E and $\langle \cdot, \cdot \rangle$ be the duality pairing between E and E'. Let $I : E \to \mathbb{R}$ be a functional and $I(u) = \Phi(u) + \Psi(u)$, where $\Phi \in C^1(E, \mathbb{R})$ and Ψ is convex and lower semicontinuous. Then the following results are given:

(i) The set $D(I) := \{u \in E \colon I(u) < +\infty\}$ is called the effective domain of I.

(ii) The sub-differential $\partial I(u)$ of the functional I at a point $u \in E$ is the following set

$$\left\{w \in E' : \left\langle \Phi'(u), v - u \right\rangle + \Psi(v) - \Psi(u) \geq \langle w, v - u \rangle,\ \forall v \in E\right\}.$$

(iii) $u \in E$ is a critical point of I such that $u \in D(I)$ and $0 \in \partial I(u)$, i.e.,

$$\left\langle \Phi'(u), v - u \right\rangle + \Psi(v) - \Psi(u) \geq 0, \quad \forall v \in E.$$

(iv) A Palais–Smale sequence at level c for I is a sequence $\{u_n\} \subset E$ such that $I(u_n) \to c$ and there is a numerical sequence $\sigma_n \to 0^+$ with

$$\left\langle \Phi'(u_n), v - u_n \right\rangle + \Psi(v) - \Psi(u_n) \geq -\sigma_n \|v - u_n\|, \quad \forall v \in E. \tag{2.6}$$

(v) The functional I satisfies the Palais–Smale condition at level c ($(PS)_c$ condition, for short) if each Palais–Smale sequence of I has a convergent subsequence in E.

To further advance the analysis, for any $u \in D(I)$, we define $I'(u) : H_c^1(\mathbb{R}^3) \to \mathbb{R}$ given by

$$\langle I'(u), z\rangle = \langle \Phi'(u), z\rangle + \int_{\mathbb{R}^3} F'(u)z \, \mathrm{d}x, \ \forall z \in H_c^1(\mathbb{R}^3)$$

and

$$\|I'(u)\| := \sup\left\{\langle I'(u), z\rangle : z \in H_c^1(\mathbb{R}^3) \text{ with } \|z\|_V \le 1\right\}.$$

If $\|I'(u)\| < +\infty$, then $I'(u)$ may be extended to a bounded operator in E_V, and it can also be considered as an element of E'_V.

From Lemma 3.3 of reference [21], we state some useful results that can help us solve the considered problem.

Lemma 2.5. *Assume that $I(u)$ satisfies (2.4). Then*

(i) if $u \in D(I)$ is a critical point of I, then for any $v \in E_V$,

$$\langle \Phi'(u), v-u\rangle + \Psi(v) - \Psi(u) \ge 0,$$

namely,

$$\int_{\mathbb{R}^3} \nabla u \nabla (v-u)\mathrm{d}x + \int_{\mathbb{R}^3} (V(x)+1)u(v-u)\mathrm{d}x - \int_{\mathbb{R}^3} \phi_u u(v-u)\,\mathrm{d}x$$
$$+ \int_{\mathbb{R}^3} F(v)\mathrm{d}x - \int_{\mathbb{R}^3} F(u)\mathrm{d}x \ge \int_{\mathbb{R}^3} G'(u)(v-u)\mathrm{d}x;$$

(ii) for every $u \in D(I)$ such that $\|I'(u)\| < +\infty$, we have $\partial I(u) \neq \emptyset$, i.e., there is $w \in E'_V$, which is denoted by $w = I'(u)$, such that for any $v \in E_V$,

$$\langle \Phi'(u), v-u\rangle + \int_{\mathbb{R}^3} F(v)\mathrm{d}x - \int_{\mathbb{R}^3} F(u)\mathrm{d}x \ge \langle w, v-u\rangle;$$

(iii) if $u \in D(I)$ is a critical point of I, then (u, ϕ_u) is one solution of system (1.7);

(iv) if $\{u_n\} \subset E_V$ is a Palais–Smale sequence, then for any $z \in H_c^1(\mathbb{R}^3)$,

$$\langle I'(u_n), z\rangle = o_n(1)\|z\|_V;$$

(v) if Λ is a bounded domain with regular boundary, then Ψ (and hence I) is of class C^1 in $H^1(\Lambda)$. Precisely, for any $u \in H^1(\Lambda)$, the functional

$$\Psi(u) = \int_\Lambda F(u)\mathrm{d}x$$

belongs to $C^1(H^1(\Lambda), \mathbb{R})$.

According to the above proprieties, we can directly get the following consequences.

Lemma 2.6. *If $u \in D(I)$ and $\|I'(u)\| < +\infty$, then $F'(u)u \in L^1(\mathbb{R}^3)$, where I satisfies (2.4).*

Proof. This proof process relies on Lemma 2.5-(ii) and (v), and for more details, readers can refer to the Lemma 2.1 in [2]. For brevity, we will omit the specifics here. □

In what follows, for each $u \in D(I)$, we can set the functional $I'(u) : E_V \to \mathbb{R}$ given by

$$I'(u)u = \int_{\mathbb{R}^3} \left(|\nabla u|^2 + (V(x)+1)u^2 - \phi_u u^2 - G'(u)u\right)\mathrm{d}x + \int_{\mathbb{R}^3} F'(u)u\,\mathrm{d}x \tag{2.7}$$
$$= \int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2 - \phi_u u^2\right)\mathrm{d}x - \int_{\mathbb{R}^3} u^2 \log u^2\,\mathrm{d}x. \tag{2.8}$$

3 Mountain pass theorem without (PS) condition

In this section, we will introduce an abstract theorem for the functional of the type $I(u) = \Phi(u) + \Psi(u)$, where $\Phi \in C^1$ and Ψ is convex and lower semicontinuous. This method was proposed by Alves and de Morais Filho [2] under the influence of Szulkin [24].

Proposition 3.1 ([2]). *Let E be a Banach space and $I\colon E \to (-\infty, +\infty]$ be a functional such that:*

(i) $I(u) = \Phi(u) + \Psi(u)$, where $\Phi(u) \in C^1(E, \mathbb{R})$, and $\Psi\colon E \to (-\infty, +\infty]$ is convex, lower semicontinuous and $\Psi(u) \not\equiv +\infty$;

(ii) $I(0) = 0$ and $I|_{\partial B_\rho(0)} \geq \alpha$, for some ρ, $\alpha > 0$;

(iii) $I(e) \leq 0$, for some $e \notin \overline{B}_\rho(0)$.

Then for fixed $\varepsilon > 0$, there is $u_\varepsilon \in E$ satisfying

$$\left\langle \Phi'\left(u_\varepsilon\right), w - u_\varepsilon \right\rangle + \Psi(w) - \Psi\left(u_\varepsilon\right) \geq -3\varepsilon \left\|w - u_\varepsilon\right\|, \quad \forall w \in E$$

and

$$I\left(u_\varepsilon\right) \in [c - \varepsilon, c + \varepsilon],$$

where

$$c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t))$$

and

$$\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, I(\gamma(1)) < 0\}.$$

Corollary 3.2. *From Proposition 3.1, it is clear that there exists a (PS) sequence $\{u_n\} \subset E$ for I, namely, $I\left(u_n\right) \to c$ and*

$$\left\langle \Phi'\left(u_n\right), w - u_n \right\rangle + \Psi(w) - \Psi\left(u_n\right) \geq -\sigma_n \left\|w - u_n\right\|, \quad \forall w \in E$$

with $\sigma_n \to 0^+$.

In the sequel, we will apply Proposition 3.1 and Corollary 3.2 to obtain our results. The most crucial point is to prove that the (PS) sequence has a convergent subsequence. First, we prove that I possesses the Mountain Pass Geometry.

Lemma 3.3. *Assume that (V_i) holds, $i = 1, 2, 3$. Then*

(i) there exist b_0, $r_0 > 0$ such that $I(u) \geq b_0$ with $\|u\|_V = r_0$;

(ii) there exists $\tilde{e} \in \mathbb{R}^3 \setminus B_{r_0}(0)$ with $\|\tilde{e}\|_V > r_0$ such that $I(\tilde{e}) < 0$.

Proof. (i) It is clear that $I(0) = 0$. According to (2.4) and (2.5) for $p \in (2, 6)$, together with $F \geq 0$ and Lemma 2.2-(4), we get

$$I(u) \geq \frac{1}{2}\|u\|_V^2 - C_1\|u\|_V^4 - C_2\|u\|_V^p \geq b_0 > 0$$

for some $b_0 > 0$ and $\|u\|_V = r_0$ small enough.

(ii) First fix $u \in D(I)\backslash\{0\}$ and $t>0$, then according to (2.3) and Lemma 2.2-(1), one can conclude

$$\begin{aligned} I(tu) &\le \frac{t^2}{2}\|u\|_V^2 - \frac{1}{2}\int_{\mathbb{R}^3} t^2u^2\log\left(|tu|^2\right)\mathrm{d}x \\ &= t^2\left(I(u)+\frac{1}{4}\int_{\mathbb{R}^3}\phi_u u^2\,\mathrm{d}x - \log t\int_{\mathbb{R}^3}u^2\,\mathrm{d}x\right) \to -\infty, \end{aligned}$$

as $t\to+\infty$. Choosing $\tilde{e}=t_*u$ with $\|\tilde{e}\|_V > r_0$ for $t_*>0$ large enough, then we obtain $I(\tilde{e})<0$. □

Remark 3.4. According to Proposition 3.1 and Lemma 3.3, there exists a (PS) sequence $\{u_n\}\subset E_V$ of $I(u)$ at the level $c>0$, where

$$c := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)) \tag{3.1}$$

and

$$\Gamma = \{\gamma\in C([0,1],E_V) : \gamma(0)=0, \gamma(1)=\tilde{e}\}.$$

Lemma 3.5. *Assume that (V_i) holds, $i=1,2,3$. If $\{u_n\}\subset E_V$ is a (PS) sequence of $I(u)$ at the level c, then $\{u_n\}$ is bounded in E_V, where c is defined in (3.1).*

Proof. By (2.3) and (2.8), for some $C>0$, we deduce

$$\begin{aligned} \int_{\mathbb{R}^3} u_n^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3}\phi_{u_n}u_n^2\,\mathrm{d}x &= 2I(u_n) - I'(u_n)u_n \\ &= 2c + o_n(1) + o_n(1)\|u_n\|_V \\ &\le C + o_n(1)\|u_n\|_V. \end{aligned}$$

Consequently,

$$\|u_n\|_2^2 \le C + o_n(1)\|u_n\|_V. \tag{3.2}$$

Next, we will use the logarithmic Sobolev inequality (1.8) for a convenient small $a>0$. Fixing $\frac{a^2}{\pi}=\frac{1}{4}$, it follows from (1.8) and (3.2) that

$$\begin{aligned} \int_{\mathbb{R}^3} u_n^2\log u_n^2\,\mathrm{d}x &\le \frac{1}{4}\|\nabla u_n\|_2^2 + \left(\log\|u_n\|_2^2 - 3\left(1+\log\sqrt{\frac{4}{\pi}}\right)\right)\|u_n\|_2^2 \\ &\le \frac{1}{4}\|\nabla u_n\|_2^2 + C_1\left(\log\|u_n\|_2^2+1\right)\|u_n\|_2^2 \\ &\le \frac{1}{4}\|\nabla u_n\|_2^2 + C_1\left(\log\left(C+o_n(1)\|u_n\|_V\right)+1\right)\left(C+o_n(1)\|u_n\|_V\right) \\ &\le \frac{1}{4}\|\nabla u_n\|_2^2 + C_2\left(\log\|u_n\|_V\right)\|u_n\|_V. \end{aligned} \tag{3.3}$$

Using the fact that given $\theta\in(0,1)$ there is $A>0$ satisfying

$$|t\log t| \le A\left(1+|t|^{1+\theta}\right), \quad t\ge 0,$$

we obtain, together with (3.3), the inequalities below

$$\begin{aligned} \int_{\mathbb{R}^3} u_n^2\log u_n^2\,\mathrm{d}x &\le \frac{1}{4}\|\nabla u_n\|_2^2 + \tilde{C}\left(1+\|u_n\|_V^{1+\theta}\right) \\ &\le \frac{1}{4}\|\nabla u_n\|_2^2 + \tilde{C}\left(1+\|u_n\|_V\right)^{1+\theta} \end{aligned} \tag{3.4}$$

for $\theta \in (0,1)$ and n large enough. Then by (3.4), we have

$$\begin{aligned} c + o_n(1)\,\|u_n\|_V &= I(u_n) - \frac{1}{4} I'(u_n)\,u_n \\ &\geq \frac{1}{4}\left(\|u_n\|_V^2 - \int_{\mathbb{R}^3} u_n^2 \log u_n^2 \,\mathrm{d}x\right) \\ &\geq C'\left(\|u_n\|_V^2 - (1 + \|u_n\|_V)^{1+\theta}\right) \end{aligned}$$

for some $C' > 0$ independent of n. From the above discussion, we get that $\{u_n\}$ is bounded in E_V. □

Remark 3.6. According to Lemma 3.3, we consider the fiber mapping $t \to f(t) := I(tu)$ given by

$$\begin{aligned} f(t) = &\frac{t^2}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + (V(x)+1)u^2\right) \mathrm{d}x \\ &- \frac{t^4}{4}\int_{\mathbb{R}^3} \phi_u u^2 \,\mathrm{d}x - \frac{t^2}{2}\int_{\mathbb{R}^3} u^2 \log|tu|^2 \,\mathrm{d}x. \end{aligned} \tag{3.5}$$

Then we can easily infer that $f(t)$ has a maximum value at the unique critical point $t_\mu > 0$. Indeed, from the expression of $f(t)$ we can observe

$$\frac{t^2}{2}\int_{\mathbb{R}^3} u^2 \log|tu|^2 \,\mathrm{d}x = \frac{t^2}{2}\int_{\mathbb{R}^3} u^2 \log|u|^2 \,\mathrm{d}x + \frac{t^2}{2}\log t^2 \int_{\mathbb{R}^3} |u|^2 \,\mathrm{d}x.$$

Hence, for given $u \in E_V$, we can denote $f(t)$ as

$$f(t) := C_1 t^2 - C_2 t^4 - C_3 t^2 \log t^2,$$

correspondingly,

$$\begin{aligned} f'(t) &= C_4 t - C_5 t^3 - C_6 t \log t^2 \\ &= t g(t), \end{aligned}$$

where

$$g(t) := C_4 - C_5 t^2 - C_6 \log t^2.$$

From the expression of g, we know that g is a monotonically decreasing function for $t > 0$, and g has the unique zero point t_μ. In other words, $g(t) > 0$ for $t \in (0, t_\mu)$ and $g(t) < 0$ for $t \in (t_\mu, +\infty)$. Hence, it's easy to conclude $f'(t) > 0$ for $t \in (0, t_\mu)$ and $f'(t) < 0$ for $t \in (t_\mu, +\infty)$, i.e., f achieves a positive maximum at the unique critical point $t_\mu > 0$.

Next, we define

$$\mathcal{M}_V = \left\{ u \in D(I) \setminus \{0\} \;\middle|\; I'(u)u = 0 \right\}. \tag{3.6}$$

In fact, for any $u \in D(I) \setminus \{0\}$, every ray $\{tu \mid t > 0\}$ intersects the set (3.6) at exactly the unique point $t_\mu u$. In this way, $t_\mu = 1$, if and only if, $u \in \mathcal{M}_V$.

Lemma 3.7. *If c denotes the mountain level associated with $I(u)$, by Remark 3.6, it is possible to prove the equality*

$$0 < c = c_V,$$

where $c_V := \inf_{u \in \mathcal{M}_V} I(u)$.

Proof. Since $u \in \mathcal{M}_V$, by Lemma 3.3 and Remark 3.6, one has

$$0 < c \leq \max_{t \geq 0} I(tu) = I(t_\mu u) = I(u),$$

namely,

$$c \leq \inf_{u \in \mathcal{M}_V} I(u).$$

Next, we need to prove the reverse inequality. Let $\{u_n\} \subset E_V$ be a $(PS)_c$ sequence of $I(u)$, then Lemma 3.5 implies that $\{u_n\}$ is bounded in E_V. Now, we claim that

$$\|u_n\|_2 \nrightarrow 0. \tag{3.7}$$

In fact, according to the contradictory argument and interpolation inequality, we have that $u_n \to 0$ in $L^p(\mathbb{R}^3)$, $\forall p \in [2,6)$. Then, by (2.5) we get

$$\int_{\mathbb{R}^3} G'(u_n)\, u_n \,\mathrm{d}x \leq C \int_{\mathbb{R}^3} |u_n|^p \,\mathrm{d}x \to 0.$$

On the other hand, from (2.7), Lemma 2.2-(4) and the fact that $F'(s)s \geq 0$, we obtain

$$\begin{aligned} \|u_n\|_V^2 + \int_{\mathbb{R}^3} F'(u_n)\, u_n \,\mathrm{d}x &= I'(u_n)u_n + \int_{\mathbb{R}^3} G'(u_n)\, u_n \,\mathrm{d}x + \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \,\mathrm{d}x \\ &= o_n(1), \end{aligned}$$

from where it follows that $u_n \to 0$ in E_V and $F'(u_n)u_n \to 0$ in $L^1(\mathbb{R}^3)$ as $n \to +\infty$. Furthermore, from the definition of $F(s)$, we can directly calculate that $0 \leq F(s) \leq F'(s)s$ for all $s \in \mathbb{R}$. Hence, $F(u_n) \to 0$ in $L^1(\mathbb{R}^3)$, and so, $I(u_n) \to 0$, which is contradictory to $I(u_n) \to c > 0$. This proves (3.7). Naturally, we may assume that there exist constants $a, b > 0$ such that

$$0 < a \leq \|u_n\|_2 \leq b, \quad \forall n \in \mathbb{N}.$$

Moreover, by Remark 3.6, for every $u_n \in E_V$, we can let $s_n > 0$ be such that $s_n u_n \in \mathcal{M}_V$. From the definition of $\mathcal{M}_V$, we can see

$$I(s_n u_n) = I(s_n u_n) - \frac{1}{2} I'(s_n u_n)\, s_n u_n, \tag{3.8}$$

which means

$$\begin{aligned} &\frac{s_n^2}{2} \int_{\mathbb{R}^3} \left(|\nabla u_n|^2 + (V(x)+1)u_n^2\right) \mathrm{d}x - \frac{s_n^4}{4} \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \,\mathrm{d}x \\ &\quad - \frac{s_n^2}{2} \int_{\mathbb{R}^3} u_n^2 \log |s_n u_n|^2 \,\mathrm{d}x = \frac{s_n^2}{2} \int_{\mathbb{R}^3} |u_n|^2 \,\mathrm{d}x + \frac{s_n^4}{4} \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \,\mathrm{d}x, \end{aligned}$$

namely,

$$\begin{aligned} &\int_{\mathbb{R}^3} \left(|\nabla u_n|^2 + V(x)u_n^2\right) \mathrm{d}x - 2 \log s_n \int_{\mathbb{R}^3} u_n^2 \,\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^3} u_n^2 \log u_n^2 \,\mathrm{d}x = s_n^2 \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \,\mathrm{d}x. \end{aligned} \tag{3.9}$$

Recalling that $\{u_n\}$ is a bounded $(PS)_c$ sequence of $I(u)$, and based on (2.8), we have

$$\begin{aligned} I'(u_n)\, u_n &= \int_{\mathbb{R}^3} \left(|\nabla u_n|^2 + V(x)u_n^2 - \phi_{u_n} u_n^2\right) \mathrm{d}x - \int_{\mathbb{R}^3} u_n^2 \log u_n^2 \,\mathrm{d}x \\ &= o_n(1). \end{aligned} \tag{3.10}$$

It follows from (3.9) and (3.10) that

$$o_n(1) = 2\|u_n\|_2^2 \log s_n + \left(s_n^2 - 1\right) \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \, \mathrm{d}x,$$

then combining (3.7) and Lemma 2.2-(4), we derive that $s_n \to 1$ as $n \to +\infty$. From this information, we arrive at

$$\inf_{u \in \mathcal{M}_V} I(u) \le I(s_n u_n) \to c \quad \text{as } n \to +\infty.$$

The proof has been completed. □

4 Proof of Theorem 1.2

Theorem 4.1 ([11])**.** *Under the condition (V_1), E_V can be compactly embedded into $L^p(\mathbb{R}^3)$, $p \in [2,6)$.*

Proof of Theorem 1.2. From Lemma 3.5, it is known that the sequence $\{u_n\}$ is bounded. Passing to a subsequence, $u_n \rightharpoonup u$ in E_V for some u and by Theorem 4.1 with (3.7), we have $u_n \to u \neq 0$ in $L^p(\mathbb{R}^3)$ for $p \in [2,6)$. Then by the Hölder inequality, combining Lemma 2.2-(4) and (2.5), respectively, we can obtain the following conclusions

$$\int_{\mathbb{R}^3} \phi_{u_n} u_n (u - u_n) \, \mathrm{d}x \to 0, \tag{4.1}$$

$$\int_{\mathbb{R}^3} G'(u_n)(u - u_n) \, \mathrm{d}x \to 0. \tag{4.2}$$

Since $\{u_n\}$ is the (PS) sequence, depending on Corollary 3.2 and taking $w = u$, we derive that

$$\langle \Phi'(u_n), u - u_n \rangle + \Psi(u) - \Psi(u_n) \ge -\sigma_n \|u - u_n\|_V,$$

namely,

$$\int_{\mathbb{R}^3} \nabla u_n \nabla (u - u_n) \, \mathrm{d}x + \int_{\mathbb{R}^3} (V(x) + 1) u_n (u - u_n) \, \mathrm{d}x - \int_{\mathbb{R}^3} \phi_{u_n} u_n (u - u_n) \, \mathrm{d}x$$
$$- \int_{\mathbb{R}^3} G'(u_n)(u - u_n) \, \mathrm{d}x + \Psi(u) - \Psi(u_n) \ge -\sigma_n \|u - u_n\|_V,$$

equivalently,

$$\langle u_n, u - u_n \rangle_V - \int_{\mathbb{R}^3} \phi_{u_n} u_n (u - u_n) \, \mathrm{d}x - \int_{\mathbb{R}^3} G'(u_n)(u - u_n) \, \mathrm{d}x$$
$$+ \Psi(u) - \Psi(u_n) \ge -\sigma_n \|u - u_n\|_V.$$

Hence, one has

$$\overline{\lim_{n \to +\infty}} \left(\langle u_n, u \rangle_V - \|u_n\|_V^2 \right) - \lim_{n \to +\infty} \left(\int_{\mathbb{R}^3} \phi_{u_n} u_n (u - u_n) \, \mathrm{d}x + \int_{\mathbb{R}^3} G'(u_n)(u - u_n) \, \mathrm{d}x \right)$$
$$+ \overline{\lim_{n \to +\infty}} \left(\Psi(u) - \Psi(u_n) \right) \ge 0.$$

Combining $u_n \rightharpoonup u$ in E_V with (4.1) and (4.2), the above inequality becomes

$$\overline{\lim_{n\to+\infty}}\left(\|u\|_V^2-\|u_n\|_V^2\right)+\overline{\lim_{n\to+\infty}}\left(\Psi(u)-\Psi\left(u_n\right)\right)\geq 0, \tag{4.3}$$

namely,

$$\|u\|_V^2-\underline{\lim_{n\to+\infty}}\|u_n\|_V^2+\Psi(u)-\underline{\lim_{n\to+\infty}}\Psi\left(u_n\right)\geq 0. \tag{4.4}$$

On the other hand, Ψ is lower semicontinuous, i.e., $\Psi(u)\leq\underline{\lim}_{n\to+\infty}\Psi\left(u_n\right)$, together with (4.4), we deduce

$$\|u\|_V^2-\underline{\lim_{n\to+\infty}}\|u_n\|_V^2\geq 0.$$

Then according to the weak lower semi-continuity of the norm $\|u\|_V^2\leq\underline{\lim}_{n\to+\infty}\|u_n\|_V^2$, we have $\|u_n\|_V\to\|u\|_V$, which implies $u_n\to u$ in E_V. Furthermore, by combining this result with Lemma 3.7, we can conclude that $I(u)=c=c_V$, i.e., u is a ground state solution for equation (1.7). The proof is completed. □

5 Proof of Theorem 1.3

In this section, we give the proofs of Theorem 1.3. Since the defined manifold $\mathcal{M}_V$ lacks C^1 regularity, we will adopt an indirect approach by borrowing the method from [17] to obtain our results.

Lemma 5.1. *Assume that (V_i) holds, $i=2,3$. If $u\in\mathcal{M}_V$ and $I(u)=c$, then u is a solution of Eq.(1.7).*

Proof. Suppose to the contrary, there exists u such that $I(u)=c$ and $I'(u)\neq 0$. Then there exists $\eta\in C_0^\infty\left(\mathbb{R}^3\right)$ such that $\langle I'(u),\eta\rangle<-1$. Choose a constant $\epsilon\in(0,1)$ small enough such that for all $|t-1|\leq\epsilon$ and $|\sigma|\leq\epsilon$,

$$\left\langle I'(tu+\sigma\eta),\eta\right\rangle\leq-\frac{1}{2}. \tag{5.1}$$

Define a cut-off function $0\leq\chi\leq 1$ such that $\chi(t)=1$ for $|t-1|\leq\frac{\epsilon}{2}$ and $\chi(t)=0$ for $|t-1|\geq\epsilon$. For $t>0$, we introduce a curve $\gamma(t)=tu$ for $|t-1|\geq\epsilon$ and $\gamma(t)=tu+\epsilon\chi(t)\eta$ for $|t-1|<\epsilon$. Clearly, $\gamma(t)$ is a continuous curve, and for $|t-1|<\epsilon$, $\|\gamma(t)\|>0$ holds when ϵ small enough. Next, we claim $I(\gamma(t))<c$, for all $t>0$. Indeed, if $|t-1|\geq\epsilon$, together with Remark 3.6, $I(\gamma(t))=I(tu)<I(u)=c$. If $|t-1|<\epsilon$, then by Lemma 2.5-(v), the mapping $[0,\epsilon]\ni\sigma\mapsto I(tu+\sigma\chi(t)\eta)$ is of C^1. Consequently, together with (5.1), there exists $\tilde{\sigma}\in(0,\epsilon)$ such that

$$I(tu+\epsilon\chi(t)\eta)=I(tu)+\left\langle I'(tu+\tilde{\sigma}\chi(t)\eta),\epsilon\chi(t)\eta\right\rangle\leq I(tu)-\frac{\epsilon}{2}\chi(t)<c.$$

Let $\omega(u)=\langle I'(u),u\rangle$. By the definition of $\gamma(t)$, we have $\omega(\gamma(1-\epsilon))=\omega((1-\epsilon)u)>0$ and $\omega(\gamma(1+\epsilon))=\omega((1+\epsilon)u)<0$. Since the mapping $t\to\omega(\gamma(t))$ is continuous, there exists $\tilde{t}\in(1-\epsilon,1+\epsilon)$ such that $\omega\left(\gamma\left(\tilde{t}\right)\right)=0$. Thus, $\gamma\left(\tilde{t}\right)\in\mathcal{M}_V$ and $I\left(\gamma\left(\tilde{t}\right)\right)<c$, which is a contradiction. The proof is complete. □

Proof of Theorem 1.3. From Proposition 3.1, Corollary 3.2 and Lemma 3.3, there exists a $(PS)_c$ sequence $\{u_n\}$ of $I(u)$ such that

$$I\left(u_n\right)\to c$$

and

$$\left\langle \Phi'\left(u_n\right), u-u_n\right\rangle+\Psi(u)-\Psi\left(u_n\right) \geq -\sigma_n\left\|u-u_n\right\|_V, \quad \forall u \in E_V,$$

with $\sigma_n \to 0^+$. As in the previous section $\{u_n\}$ is a bounded sequence in E_V, then there is $u_0 \in E_V$, and a subsequence of $\{u_n\}$, still denoted $\{u_n\}$, such that

$$u_n \rightharpoonup u_0 \quad \text{in } E_V, \tag{5.2}$$

$$u_n \to u_0 \quad \text{in } L^2_{\text{loc}}\left(\mathbb{R}^3\right), \tag{5.3}$$

$$u_n \to u_0 \quad \text{a.e. in } \mathbb{R}^3. \tag{5.4}$$

Next, taking any test function $\eta \in C_0^\infty(\mathbb{R}^3)$, from (2.5) and (5.2) together with Lemmas 2.2-(4) and 2.5-(v), we have

$$\begin{aligned} 0 &= \lim_{n\to+\infty}\left\langle I'\left(u_n\right), \eta\right\rangle \\ &= \lim_{n\to+\infty}\left[\langle u_n, \eta\rangle_V - \int_{\mathbb{R}^3}\phi_{u_n}u_n\eta \,\mathrm{d}x + \int_{\mathbb{R}^3}F'(u_n)\eta\,\mathrm{d}x - \int_{\mathbb{R}^3}G'(u_n)\eta\,\mathrm{d}x\right] \\ &= \langle u_0, \eta\rangle_V - \int_{\mathbb{R}^3}\phi_{u_0}u_0\eta\,\mathrm{d}x + \int_{\mathbb{R}^3}F'(u_0)\eta\,\mathrm{d}x - \int_{\mathbb{R}^3}G'(u_0)\eta\,\mathrm{d}x \\ &= \left\langle I'\left(u_0\right), \eta\right\rangle, \end{aligned}$$

which means that u_0 is a weak solution to equation (1.7). To complete this proof of Theorem 1.3, the key is to prove that $u_0 \neq 0$ in E_V. In fact, combined with (3.7), the Lions Concentration Compactness Principle [16] implies that there are parameters r, $\beta > 0$, and a sequence $\{y_n\} \subset \mathbb{Z}^3$ such that

$$\lim_{n\to+\infty}\int_{B_r(y_n)}|u_n|^2\,\mathrm{d}x \geq \beta > 0.$$

Now, setting $v_n(x) = u_n\left(x+y_n\right)$, $\{y_n\} \subset \mathbb{Z}^3$, it follows that

$$\int_{B_r(0)}|v_n|^2\,\mathrm{d}x = \int_{B_r(0)}\left|u_n\left(x+y_n\right)\right|^2\,\mathrm{d}x = \int_{B_r(y_n)}|u_n|^2\,\mathrm{d}x \geq \frac{\beta}{2} > 0. \tag{5.5}$$

Since $V(x)$ satisfies (V_2), there hold $\|v_n\|_V = \|u_n\|_V$ and

$$I\left(v_n\right) \to c,\ I'\left(v_n\right) \to 0, \quad \text{as } n \to +\infty,$$

so $\{v_n\}$ is also a bounded $(PS)_c$ sequence of I. Therefore, if v_0 denotes the weak limit of $\{v_n\}$ in E_V, for some subsequence, we have that

$$v_n \rightharpoonup v_0 \quad \text{in } E_V, \tag{5.6}$$

$$v_n \to v_0 \quad \text{a.e. in } \mathbb{R}^3, \tag{5.7}$$

$$v_n \to v_0 \quad \text{in } L^q\left(B_r(0)\right), \quad \forall\, r>0 \text{ and } q \in [1,6). \tag{5.8}$$

From (5.8), we derive that

$$\int_{B_r(0)}|v_0|^2\,\mathrm{d}x \geq \frac{\beta}{2} > 0,$$

which shows that $v_0 \neq 0$. Using the standard argument, one has $I'(v_0)\eta = 0$, $\forall \eta \in C_0^\infty\left(\mathbb{R}^3\right)$, i.e., v_0 is a non trivial weak solution for equation (1.7).

Further, because $v_0 \in \mathcal{M}_V$, it follows from (2.3), (2.8) and the Fatou lemma that

$$\begin{aligned} 2c \leq 2I(v_0) &= 2I(v_0) - I'(v_0)v_0 \\ &= \int_{\mathbb{R}^3} {v_0}^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} \phi_{v_0}{v_0}^2 \,\mathrm{d}x \\ &\leq \underline{\lim}_{n\to+\infty} \left(\int_{\mathbb{R}^3} v_n^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} \phi_{v_n} v_n^2 \,\mathrm{d}x \right) \\ &\leq \overline{\lim}_{n\to+\infty} \left(\int_{\mathbb{R}^3} v_n^2 \,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} \phi_{v_n} v_n^2 \,\mathrm{d}x \right) \\ &= \overline{\lim}_{n\to+\infty} \left(2I(v_n) - I'(v_n)v_n\right) = 2c, \end{aligned}$$

that is, $c_V = c = I(v_0)$ and consequently, v_0 is a ground state solution of equation (1.7). □

6 Proof of Theorem 1.4

In this section, we modify some notations.

Consider the vector space $H^1(\mathbb{R}^3)$ endowed with the norm

$$\|u\|_H = \left(\int_{\mathbb{R}^3} \left(|\nabla u|^2 + (V_p(x)+1)\, u^2 \right) \mathrm{d}x \right)^{\frac{1}{2}},$$

where V_p is defined in (V_3). By replacing V by V_p, we have a periodic problem as in the following problem

$$\begin{cases} -\Delta u + V_p(x)u - \phi u = u \log u^2 & \text{in } \mathbb{R}^3, \\ -\Delta\phi + \Delta^2\phi = 4\pi u^2 & \text{in } \mathbb{R}^3. \end{cases} \tag{6.1}$$

The underlying energy functional $I_p : H^1(\mathbb{R}^3) \to (-\infty, +\infty]$ associated with problem (6.1) can be defined as

$$I_p(u) = \Phi_p(u) + \Psi(u), \quad \forall u \in H^1(\mathbb{R}^3), \tag{6.2}$$

where

$$\Phi_p(u) := \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + (V_p(x)+1)\, u^2 \right) \mathrm{d}x - \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 \,\mathrm{d}x - \int_{\mathbb{R}^3} G(u)\,\mathrm{d}x,$$

and

$$\Psi(u) := \int_{\mathbb{R}^3} F(u)\,\mathrm{d}x.$$

Remark 6.1. Note that we would like to point out that the proof of Lemma 3.7 is independent of potential conditions. Therefore, in the case of asymptotically periodic potentials, the following items are valid:

(i) if c denotes the mountain level associated with $I(u)$, we have

$$0 < c = c_V := \inf_{u\in\mathcal{M}_V} I(u), \tag{6.3}$$

where $\mathcal{M}_V$ is defined in (3.6).

(ii) If d denotes the mountain level associated with $I_p(u)$, we have that

$$0 < d = c_{V_p} := \inf_{u \in \mathcal{M}_p} I_p(u), \tag{6.4}$$

where

$$\mathcal{M}_p = \left\{ u \in D(I) \setminus \{0\} : I_p'(u)u = 0 \right\}.$$

Lemma 6.2. *Assume that $V(x)$ satisfies (V_3). Then*

(i) $c_V < c_{V_p}$.

(ii) If $I(u_n) \to c \in (0, c_{V_p})$ and $I'(u_n) \to 0$, then $u_n \rightharpoonup u_0 \neq 0$ after passing to a subsequence, u_0 is a critical point of $I(u)$ and $I(u_0) \le c$.

Proof. (i) Similar to Remark 3.6, we know that for any $u \in \mathcal{M}_p$, one has $I_p(u) = \max_{t>0} I_p(tu)$, and there exists $t_\mu > 0$ such that $t_\mu u \in \mathcal{M}_V$ and $I(t_\mu u) = \max_{t>0} I(tu)$. By (V_3),

$$c_V = \inf_{u \in \mathcal{M}_V} I(u) \le I(t_\mu u) = \max_{t>0} I(tu) < \max_{t>0} I_p(tu) = I_p(u).$$

Because of the arbitrariness of u, we get that $c_V < c_{V_p}$.

(ii) As in Lemma 3.5, $\{u_n\}$ is bounded in E_V with the level c. Up to a subsequence, we can assume that

$$\begin{aligned} u_n &\rightharpoonup u_0 \quad \text{in } E_V, \\ u_n &\to u_0 \quad \text{in } L^2_{\text{loc}}\left(\mathbb{R}^3\right), \\ u_n &\to u_0 \quad \text{a.e. in } \mathbb{R}^3. \end{aligned}$$

Similar to the proof of Theorem 1.3, taking any $\varphi \in C_0^\infty(\mathbb{R}^3)$, one has $I'(u_0)\varphi = 0$, i.e., u_0 is a weak solution of equation (1.7). By Fatou's lemma, one has

$$\begin{aligned} I(u_0) = I(u_0) - \frac{1}{2} I'(u_0)u_0 &= \int_{\mathbb{R}^3} {u_0}^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^3} \phi_{u_0} {u_0}^2 \, dx \\ &\le \varliminf_{n \to +\infty} \left(\int_{\mathbb{R}^3} u_n^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \, dx \right) \\ &= \varliminf_{n \to +\infty} \left(I(u_n) - \frac{1}{2} I'(u_n)u_n \right) = c, \end{aligned} \tag{6.5}$$

so, $I(u_0) \le c$. Next, we claim that $u_0 \neq 0$ in E_V. Suppose, by contradiction, that $u_0 = 0$. For any $\epsilon > 0$, there exists $R(\epsilon) > 0$ such that

$$\left| V(x) - V_p(x) \right| < \epsilon, \quad \forall |x| > R.$$

Since $u_n \to u_0 = 0$ in $L^2_{\text{loc}}\left(\mathbb{R}^3\right)$ and $\{u_n\}$ is bounded, we obtain

$$\int_{\mathbb{R}^3} \left| V(x) - V_p(x) \right| u_n^2 \, dx \le \int_{B_R(0)} \left| V(x) - V_p(x) \right| u_n^2 \, dx + \epsilon \int_{\mathbb{R}^3 \setminus B_R(0)} u_n^2 \, dx = o_n(1),$$

which yields, as $n \to +\infty$

$$I(u_n) - I_p(u_n) = \frac{1}{2} \int_{\mathbb{R}^3} \left| V(x) - V_p(x) \right| u_n^2 \, dx \to 0.$$

Using the Hölder inequality and taking w with $\|w\| = 1$, we obtain, as $n \to +\infty$

$$\begin{aligned}\left|\left\langle I'(u_n) - I'_p(u_n), w\right\rangle\right| &\leq \int_{\mathbb{R}^3} \left|V(x) - V_p(x)\right| |u_n| |w| \, dx \\ &\leq C \left(\int_{\mathbb{R}^3} \left|V(x) - V_p(x)\right| u_n^2 \, dx \right)^{\frac{1}{2}} \\ &\to 0.\end{aligned}$$

In summary, we can immediately conclude

$$I_p(u_n) \to c \quad \text{and} \quad I'_p(u_n) \to 0,$$

namely, $\{u_n\}$ is a bounded $(PS)_c$ sequence for I_p. From (3.7) and Lions lemma [16], it follows that there exist $R, \kappa > 0$, $\{z_n\} \subset \mathbb{Z}^3$ satisfying

$$\int_{B_R(z_n)} |u_n|^2 \, dx \geq \kappa > 0, \tag{6.6}$$

for all $n \in \mathbb{N}$. Since $u(x) = 0$ for all $x \in \mathbb{R}^3$, we have $|z_n| \to +\infty$. Taking $\omega_n(x) = u_n(x + z_n)$, then according to condition (V_3), $\{\omega_n\}$ is also a bounded $(PS)_c$ sequence for I_p. Therefore, we can assume there exists $\omega_0 \in E_V$ such that $\omega_n \rightharpoonup \omega_0$ in E_V, $\omega_n \to \omega_0$ in $L^2_{\text{loc}}(\mathbb{R}^3)$ and $\omega_n \to \omega_0$ a.e. in $\mathbb{R}^3$ up to a subsequence. By (6.6), we have

$$\int_{B_R(0)} |\omega_n|^2 \, dx = \int_{B_R(z_n)} |u_n|^2 \, dx \geq \kappa > 0,$$

so, $\omega_0 \neq 0$. Then, using the standard argument, we get that for any $v \in C_0^\infty(\mathbb{R}^3)$, $I'_p(\omega_0) v = 0$, i.e., ω_0 is a nontrivial week solution of Eq. (6.1). Following the proof of Theorem 1.3, we know that ω_0 is a ground-state solution for equation (6.1), i.e.,

$$c_{V_p} = I_p(\omega_0). \tag{6.7}$$

By repeating the method of (6.5), this inequality $I_p(\omega_0) \leq c$ holds. Therefore, together with (6.7) and the fact that $c \in (0, c_{V_p})$, we obtain the following conclusions

$$I_p(\omega_0) \leq c < c_{V_p} = I_p(\omega_0),$$

which is a contradiction. This completes the proof. □

Proof of Theorem 1.4. From Lemmas 3.3 and 3.5, there exists a bounded (PS) sequence $\{u_n\}$ for I with the level $c \in (0, c_{V_p})$. By Lemma 6.2-(ii), we obtain a critical point $u_0 \neq 0$ of I such that $I(u_0) \leq c$. So, we have $I'(u_0)u_0 = 0$, i.e., $u_0 \in \mathcal{M}_V$. Then, arguing again as in the periodic case, it is possible to prove that u_0 is a ground state solution for Eq.(1.7). This completes the proof of Theorem 1.4. □

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