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Combined effects of singular attractive and asymptotically linear terms in a Kirchhoff Dirichlet problem

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Abstract. In this paper, we consider the following Kirchhoff Dirichlet problem with singular attractive term

$$\begin{cases} -\left(a+b\displaystyle\int_\Omega |\nabla u|^2 dx\right)\Delta u = u^{-\gamma}+\lambda f(u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 3)$, the parameters $a, b, \lambda, \gamma > 0$ and $f(u)$ is asymptotically linear reaction. In particular, we investigate both the strong singular case $(\gamma \geq 1)$ and the weak singular case $(0 < \gamma < 1)$ employing different techniques to reflect the distinct nature of each scenario. In the first case, ground state solutions are obtained via a direct minimizing methods, while in the latter case we combine variational theory with perturbation methods to prove the existence of ground state solutions for above Kirchhoff problem.

Keywords: positive solution, Kirchhoff Dirichlet problem, singularity, asymptotically linear reaction.

2020 Mathematics Subject Classification: 35B25, 35J60.

1 Introduction

Considering the following Kirchhoff Dirichlet problem

$$\begin{cases} -\left(a+b\displaystyle\int_\Omega |\nabla u|^2 dx\right)\Delta u = f(x,u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 3)$, the parameters $a, b \geq 0$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function. The equation (1.1) was first introduced by Kirchhoff [13] as

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an extension of the classical D'Alembert's wave equations for free vibration of elastic strings. It is worth mentioning that fundamental analysis approach was introduced by Lions in the pioneer work [21]. After then, the existence results for above equations have been widely investigated, we refer the reader to [5,6,9–12,14,18,19,23–26,30,31] and the references therein.

Very recently, there are some papers on the Kirchhoff Dirichlet problem involving singularity, see [15,16,20,22,27,28] and the references therein. For example, in [28], Wang and Yan consider a class of Kirchhoff Dirichlet problem with singularity and nonlinearity in a bounded domain:

$$\begin{cases} -\left(a+b\int_\Omega |\nabla u|^2dx\right)\Delta u = \dfrac{f(x)}{u} - \mu u^p & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases}$$

where $1<p<2^*-1$, $2^*=\frac{2N}{N-2}$, Ω is a smooth bounded domain in $\mathbb{R}^N (N\geq 3)$, the function $f(x)\in L^2(\Omega)$ is positive, and a,b,μ are real numbers. They using the approximation method, a unique positive solution is obtained.

In particular, in [20,22], the following Kirchhoff Dirichlet problem with singularity and Sobolev subcritical perturbation is considered:

$$\begin{cases} -\left(a+b\int_\Omega |\nabla u|^2dx\right)\Delta u = u^p + \dfrac{\lambda}{u^\gamma} & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases}$$

where $0<p<5$, $0<\gamma<1$, and existence of solutions was obtained via using variational methods. Later, Lei, Liao and Tang [16] extended main results in [20,22] from Sobolev subcritical case to Sobolev critical case $(p=5)$:

$$\begin{cases} -\left(a+b\int_\Omega |\nabla u|^2dx\right)\Delta u = u^5 + \dfrac{\lambda}{u^\gamma} & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^3$, and λ is a real parameter, $\gamma\in(0,1)$ is a constant. Such problem may occur in various branches of mathematical physics. The authors obtain two positive solutions via the variational and perturbation methods. Moreover, in [15], Lei and Liao studied the multiplicity of solutions for above Kirchhoff Dirichlet problem with more general nonlinearity $f(x,u)$, instead of power term u^p. In addition, many scholars are also interested in the p-Laplacian type Kirchhoff equation with critical reactions [3,17].

To the best of our knowledge, for strong singular case $\gamma\geq 1$, the existence results of ground state solution has not been studied up to now. The study of ground state solutions was started in the seminal work of Berestycki and Lions [4] and Coleman, Glazer and Martin [8]. In the calculus of variations, the ground state solution is the function that minimizes a given energy functional. Thus in this paper we study the ground state solutions of Kirchhoff Dirichlet problem involving singular and asymptotically linear terms:

$$\begin{cases} -\left(a+b\int_\Omega |\nabla u|^2dx\right)\Delta u = u^{-\gamma} + \lambda f(u) & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N (N\geq 3)$, the parameters $a,b,\lambda,\gamma>0$. The continuous function $f:\mathbb{R}^+\mapsto\mathbb{R}^+, \mathbb{R}^+=[0,+\infty)$ satisfies

(f_1) $\lim_{s\to\infty} \frac{f(s)}{s} = \theta$, for some $\theta \in (0, +\infty)$;

(f_2) the function $s \mapsto \frac{f(s)}{s}$ is non-increasing in $(0, +\infty)$.

The function $u(x)$ is called a weak solution of (1.2) if $0 < u(x) \in H_0^1(\Omega)$ and for all $\varphi \in H_0^1(\Omega)$ it holds

$$(a + b\|u\|^2) \int_\Omega \nabla u \nabla \varphi dx - \lambda \int_\Omega f(u)\varphi dx - \int_\Omega u^{-\gamma}\varphi dx = 0.$$

Define the function $G(t)$ as follows:

- for $0 < \gamma < 1$, $G(t) = \frac{(t^+)^{1-\gamma}}{1-\gamma}$, $t \in \mathbb{R}$;
- for $\gamma = 1$, $G(t) = \begin{cases} \ln t, & t > 0, \\ +\infty, & t = 0; \end{cases}$ and for $\gamma > 1$, $G(t) = \begin{cases} \frac{t^{1-\gamma}}{1-\gamma}, & t > 0, \\ +\infty, & t = 0. \end{cases}$

Let

$$f_0(t) = \begin{cases} f(t), & t \geq 0, \\ f(0), & t < 0, \end{cases} \quad \text{and} \quad F(t) = \int_0^t f_0(s)ds.$$

Denote by δ_1 the first eigenvalue of $\left(-\Delta, H_0^1(\Omega)\right)$ [29] and set

$$\lambda_* = \frac{a\delta_1}{\theta}. \tag{1.3}$$

Now the main results can be described as follows.

Theorem 1.1. *Assume that $\gamma \geq 1$, the conditions (f_1)-(f_2) hold and*

$$D = \left\{ u \in H_0^1(\Omega) : \int_\Omega G(|u|)dx \in \mathbb{R} \right\} \neq \emptyset.$$

Then for every $\lambda \in (0, \lambda_*)$, there exists a positive ground state solution u_λ of (1.2). Moreover, u_λ is a global minimizer of the action functional I_λ, where

$$I_\lambda(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \lambda \int_\Omega F(u)dx - \int_\Omega G(|u|)dx.$$

Theorem 1.2. *Assume that $0 < \gamma < 1$ and the conditions (f_1)–(f_2) hold. Then for every $\lambda \in (0, \lambda_*)$, there exists a positive ground state solution $\tilde{u}_\lambda$ of (1.2). Moreover, $\tilde{u}_\lambda$ is a global minimizer of I_λ.*

Remark 1.3. (i) The set D in Theorem 1.1 is not closed as usual, see [1]. (ii) Theorems 1.1 extend main results in [15] from weak singular case $0 < \gamma < 1$ to strong singular case $\gamma \geq 1$. Moreover, following the ideas in [1,2], the methods in Theorem 1.2 to obtain global minimizer are more direct.

Notations: Throughout this paper, we make use of the following notations:

- The space $H_0^1(\Omega)$ is equipped with the norm $\|u\|^2 = \int_\Omega |\nabla u|^2 dx$, the norm in $L^p(\Omega)$ is denoted by $|\cdot|_p$;
- If A is a measurable set in $\mathbb{R}^N$, we denote by $|A|$ the Lebesgue measure of A;
- $C, C_1, C_2, C_3, \ldots,$ denote different positive constants ;

- Denote by d the distance function to the boundary;
- $u^+ = \max\{0, u\}$, $u_- = \max\{0, -u\}$.

The remainder of this paper is organized as follows. In Section 2, we consider the strong singular case and give the proof of Theorem 1.1. We consider the weak singular case and give the proof of Theorem 1.2 in Section 3.

2 The strong singular case: $\gamma \geq 1$

Since we are interested in positive solutions, we introduce the set

$$D^+ = \{u \in D : u \geq 0 \text{ a.e.} \quad \text{in } \Omega\}.$$

Now we show that $\inf_{u\in D} I_\lambda(u) = \inf_{u\in D^+} I_\lambda(u)$. In fact, it is easy to see that $\inf_{u\in D} I_\lambda(u) \leq \inf_{u\in D^+} I_\lambda(u)$. On the other hand, we have

$$\begin{aligned} I_\lambda(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \lambda \int_\Omega F(u)dx - \int_\Omega G(|u|)dx \\ &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \lambda \int_{u<0} f(0)udx - \lambda \int_{u\geq 0} F(u)dx - \int_\Omega G(|u|)dx \\ &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \lambda \int_\Omega F(|u|)dx - \int_\Omega G(|u|)dx \\ &= I_\lambda(|u|) \geq \inf_{u\in D^+} I_\lambda(u), \quad \forall\, u \in D. \end{aligned}$$

Hence, $\inf_{u\in D} I_\lambda(u) = \inf_{u\in D^+} I_\lambda(u)$. This implies that a global minimum for $I_\lambda(u)$ exists if and only if there exists $v \in D^+$, such that $I_\lambda(v) = \inf_{u\in D^+} I_\lambda(u)$.

Now, we are in position to state the following lemma, which provides the existence of a global minimizer of $I_\lambda(u)$ for every $\lambda \in (0, \lambda_*)$.

Lemma 2.1. *Assume $\lambda \in (0, \lambda_*)$. Then, there exists $u_\lambda \in D^+$, such that*

$$I_\lambda(u_\lambda) = \inf_{u\in D^+} I_\lambda(u).$$

Proof. Take $\varepsilon > 0$, satisfies $\frac{\lambda(\theta+\varepsilon)}{a\delta_1} < 1$. From (f_1)–(f_2), for $\forall\, \varepsilon > 0$, there exist C_1, C_2, such that

$$\theta s \leq f(s) \leq (\theta + \varepsilon)s + C_1, \quad \forall\, s \geq 0, \tag{2.1}$$

$$\theta \frac{s^2}{2} \leq F(s) \leq \frac{\theta + \varepsilon}{2} s^2 + C_2 s, \quad \forall\, s \geq 0. \tag{2.2}$$

By (2.2), Sobolev embedding and Poincaré inequality, it follows that

$$\begin{aligned} I_\lambda(u) &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \lambda \int_\Omega \left(\frac{\theta+\varepsilon}{2}u^2 + C_2 u\right) dx - \int_\Omega G(|u|)dx \\ &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\lambda(\theta+\varepsilon)}{2}\frac{\|u\|^2}{\delta_1} - C_3\|u\| - \int_\Omega G(|u|)dx \\ &= \frac{a}{2}\left(1 - \frac{\lambda(\theta+\varepsilon)}{a\delta_1}\right)\|u\|^2 + \frac{b}{4}\|u\|^4 - C_3\|u\| - \int_\Omega G(|u|)dx, \quad \forall\, u \in D^+. \end{aligned} \tag{2.3}$$

Now, we consider the following two cases:
Case i: $\gamma = 1$. By the inequality $\ln|u| \le |u|$, (2.3) and Sobolev embedding, we deduce that

$$\begin{aligned} I_\lambda(u) &\ge \frac{a}{2}\left(1 - \frac{\lambda(\theta+\varepsilon)}{a\delta_1}\right)\|u\|^2 + \frac{b}{4}\|u\|^4 - C_3\|u\| - \int_\Omega \ln|u|dx \\ &\ge \frac{a}{2}\left(1 - \frac{\lambda(\theta+\varepsilon)}{a\delta_1}\right)\|u\|^2 + \frac{b}{4}\|u\|^4 - C_3\|u\| - \int_\Omega |u|dx \\ &\ge \frac{a}{2}\left(1 - \frac{\lambda(\theta+\varepsilon)}{a\delta_1}\right)\|u\|^2 + \frac{b}{4}\|u\|^4 - C_4\|u\|, \end{aligned}$$

which implies that $I_\lambda(u)$ is coercive in D^+. Let $\{u_n\} \subset D^+$ be a minimizing sequence, such that $I_\lambda(u_n) \to \inf_{u\in D^+} I_\lambda(u)$ as $n \to \infty$. Therefore, we may assume that $u_\lambda \in H_0^1(\Omega)$, such that

$$\begin{cases} u_n \rightharpoonup u_\lambda & \text{in } H_0^1(\Omega), \\ u_n \to u_\lambda & \text{in } L^p(\Omega), \ \ p \in (0,6), \\ u_n \to u_\lambda & \text{a.e. in } \Omega. \end{cases} \tag{2.4}$$

Since $\{u_n\} \subset D^+$ one has $u_n \ge 0$ a.e. in Ω, thus $u_\lambda \ge 0$ a.e. in Ω. Moreover, since $u_n \to u_\lambda$ in $L^1(\Omega)$ there exists $h \in L^1(\Omega)$ such that $|u_n| \le h$ a.e. in Ω. So, $\ln|u_n| \le |u_n| \le h$ a.e. in Ω, applying the Reverse Fatou's lemma, we have

$$-\infty < \limsup_{n\to\infty} \int_\Omega \ln|u_n|dx \le \int_\Omega \limsup_{n\to\infty} \ln|u_n|dx = \int_\Omega \ln|u_\lambda|dx. \tag{2.5}$$

Note that if $u_\lambda = 0$ in a set of positive measure A, then $\int_\Omega \ln|u_\lambda|dx = \int_{\Omega\setminus A} \ln|u_\lambda|dx + \int_A \ln|u_\lambda|dx \to -\infty$, because $\ln|u_\lambda| = -\infty$ in A, it contradicts (2.5). Therefore, we obtain $u_\lambda > 0$ in Ω. Next, we show that the sequence $\{\int_\Omega G(|u_n|)dx\} = \{\int_\Omega \ln|u_n|dx\}$ is bounded, if not, we should have $-\int_\Omega \ln|u_n|dx \to +\infty$, then $\liminf_{n\to\infty} I_\lambda(u_n) = +\infty$, which is a contradiction with $\inf_{u\in D^+} I_\lambda(u) = \liminf_{n\to\infty} I_\lambda(u_n) \in \mathbb{R}$. This shows that $\{\int_\Omega G(|u_n|)dx\} = \{\int_\Omega \ln|u_n|dx\}$ is bounded. Then, we have $u_\lambda \in D^+$ and

$$\begin{aligned} \inf_{u\in D^+} I_\lambda(u) &= \liminf_{n\to\infty} I_\lambda(u_n) \\ &\ge \frac{a}{2}\|u_\lambda\|^2 + \frac{b}{4}\|u_\lambda\|^4 - \lambda\int_\Omega F(u_\lambda)dx - \limsup_{n\to\infty}\int \ln|u_n|dx \\ &\ge \frac{a}{2}\|u_\lambda\|^2 + \frac{b}{4}\|u_\lambda\|^4 - \lambda\int_\Omega F(u_\lambda)dx - \int_\Omega \ln|u_\lambda|dx = I_\lambda(u_\lambda) \ge \inf_{u\in D^+} I_\lambda(u). \end{aligned}$$

Thus, $I_\lambda(u_\lambda) = \inf_{u\in D^+} I_\lambda(u)$.
Case ii: $\gamma > 1$. From (2.3), we have

$$\begin{aligned} I_\lambda(u) &\ge \frac{a}{2}\left(1 - \frac{\lambda(\theta+\varepsilon)}{a\delta_1}\right)\|u\|^2 + \frac{b}{4}\|u\|^4 - C_3\|u\| - \frac{1}{1-\gamma}\int_\Omega |u|^{1-\gamma}dx \\ &> \frac{a}{2}\left(1 - \frac{\lambda(\theta+\varepsilon)}{a\delta_1}\right)\|u\|^2 + \frac{b}{4}\|u\|^4 - C_3\|u\|, \quad \forall u \in D^+, \end{aligned}$$

which implies that $I_\lambda(u)$ is coercive in D^+. Let $\{u_n\} \subset D^+$ be a minimizing sequence, such that $I_\lambda(u_n) \to \inf_{u\in D^+} I_\lambda(u)$ as $n \to \infty$. Therefore, we may assume that $0 \le u_\lambda \in H_0^1(\Omega)$, such that (2.4) hold. Similar with Case i, we have $u_\lambda > 0$ and the sequence $\{\int_\Omega G(|u_n|)dx\} = \{\int_\Omega |u_n|^{1-\gamma}dx\}$ is bounded. By Fatou's lemma, one has

$$-\infty < \limsup_{n\to\infty} \int_\Omega \frac{1}{1-\gamma}|u_n|^{1-\gamma}dx \le \int_\Omega \limsup_{n\to\infty} \frac{1}{1-\gamma}|u_n|^{1-\gamma}dx = \frac{1}{1-\gamma}\int_\Omega |u_\lambda|^{1-\gamma}dx,$$

which implies that $u_\lambda \in D^+$ and

$$\begin{aligned}\inf_{u\in D^+} I_\lambda(u) &= \liminf_{n\to\infty} I_\lambda(u_n)\\ &\geq \frac{a}{2}\|u_\lambda\|^2 + \frac{b}{4}\|u_\lambda\|^4 - \lambda\int_\Omega F(u_\lambda)dx - \limsup_{n\to\infty}\frac{1}{1-\gamma}\int_\Omega |u_n|^{1-\gamma}dx\\ &\geq \frac{a}{2}\|u_\lambda\|^2 + \frac{b}{4}\|u_\lambda\|^4 - \lambda\int_\Omega F(u_\lambda)dx - \int_\Omega \frac{1}{1-\gamma}|u_\lambda|^{1-\gamma}dx\\ &= I_\lambda(u_\lambda) \geq \inf_{u\in D^+} I_\lambda(u).\end{aligned}$$

Thus, $I_\lambda(u_\lambda) = \inf_{u\in D^+} I_\lambda(u)$. □

Proof of Theorem 1.1. Let u_λ be as in Lemma 2.1, we prove that u_λ is a solution of Eq. (1.2). For arbitrary $\varepsilon > 0$ and $\phi \in H_0^1(\Omega),\ \phi \geq 0\ in\ \Omega$, to show that $u_\lambda + \varepsilon\phi \in D^+$, we divide the proof into two cases.

Case 1. $\gamma = 1$. Since $\ln(u) \leq u$ for all $u > 0$, and $u_\lambda \in D^+ \subset D$, then

$$-\infty < \int_\Omega \ln(u_\lambda) \leq \int_\Omega \ln(u_\lambda + \varepsilon\phi) \leq \int_\Omega (u_\lambda + \varepsilon\phi) < +\infty,$$

that is, $u_\lambda + \varepsilon\phi \in D^+$.

Case 2. $\gamma > 1$. Since $\gamma > 1, u_\lambda + \varepsilon\phi \geq u_\lambda > 0$, then

$$\int_\Omega |u_\lambda + \varepsilon\phi|^{1-\gamma}dx \leq \int_\Omega |u_\lambda|^{1-\gamma}dx < +\infty,$$

that is, $u_\lambda + \varepsilon\phi \in D^+$.

Thus, for $\gamma \geq 1$, we have

$$I_\lambda(u_\lambda + \varepsilon\phi) \geq I_\lambda(u_\lambda). \tag{2.6}$$

It follows from (2.6) that

$$\begin{aligned}&\frac{a}{2}(\|u_\lambda + \varepsilon\phi\|^2 - \|u_\lambda\|^2) - \lambda\int_\Omega (F(u_\lambda + \varepsilon\phi - F(u_\lambda))dx)\\ &+\frac{b}{4}(\|u_\lambda + \varepsilon\phi\|^4 - \|u_\lambda\|^4) - \int_\Omega (G(u_\lambda + \varepsilon\phi) - G(u_\lambda))dx \geq 0.\end{aligned} \tag{2.7}$$

Dividing by $\varepsilon > 0$ and passing to the limit as $\varepsilon \to 0^+$ in (2.7), it gives

$$a\int_\Omega \nabla u_\lambda \nabla\phi dx + b\|u_\lambda\|^2\int_\Omega \nabla u_\lambda\nabla\phi dx - \lambda\int_\Omega f(u_\lambda)\phi dx - \int_\Omega u_\lambda^{-\gamma}\phi dx \geq 0. \tag{2.8}$$

Suppose for any $t > 0$, $tu_\lambda \in D^+$. Define $\psi \in C^1((0, +\infty), \mathbb{R})$ by $\psi(t) = I_\lambda(tu_\lambda)$. Obviously, ψ has a global minimum at $t = 1$, and so

$$0 = \psi'(1) = a\|u_\lambda\|^2 + b\|u_\lambda\|^4 - \lambda\int_\Omega f(u_\lambda)u_\lambda dx - \int_\Omega u_\lambda^{1-\gamma}dx. \tag{2.9}$$

Set $\Psi(x) = (u_\lambda(x) + \varepsilon\phi(x))^+$, $\forall\, \phi \in H_0^1(\Omega)$, $\varepsilon > 0$. From (2.8) and (2.9), it follows that

$$\begin{aligned}
0 &\le a\int_\Omega \nabla u_\lambda \nabla \Psi dx + b\|u_\lambda\|^2 \int_\Omega \nabla u_\lambda \nabla \Psi dx - \lambda \int_\Omega f(u_\lambda)\Psi dx - \int_\Omega u_\lambda^{-\gamma}\Psi dx \\
&= \int_{u_\lambda+\varepsilon\phi\ge 0} (a+b\|u_\lambda\|^2)\nabla u_\lambda \nabla(u_\lambda+\varepsilon\phi)dx \\
&\quad - \lambda\int_{u_\lambda+\varepsilon\phi\ge 0} f(u_\lambda)(u_\lambda+\varepsilon\phi)dx - \int_{u_\lambda+\varepsilon\phi\ge 0} u_\lambda^{-\gamma}(u_\lambda+\varepsilon\phi)dx \\
&= \int_\Omega [(a+b\|u_\lambda\|^2)\nabla u_\lambda\nabla(u_\lambda+\varepsilon\phi) - \lambda f(u_\lambda)(u_\lambda+\varepsilon\phi) - u_\lambda^{-\gamma}(u_\lambda+\varepsilon\phi)]dx \\
&\quad - \int_{u_\lambda+\varepsilon\phi<0} [(a+b\|u_\lambda\|^2)\nabla u_\lambda\nabla(u_\lambda+\varepsilon\phi) - \lambda f(u_\lambda)(u_\lambda+\varepsilon\phi) - u_\lambda^{-\gamma}(u_\lambda+\varepsilon\phi)]dx \qquad (2.10)\\
&\le \psi'(1) + \varepsilon\int_\Omega [(a+b\|u_\lambda\|^2)\nabla u_\lambda\nabla\phi - \lambda f(u_\lambda)\phi - u_\lambda^{-\gamma}\phi]dx \\
&\quad - \varepsilon\int_{u_\lambda+\varepsilon\phi<0} (a+b\|u_\lambda\|^2)\nabla u_\lambda\nabla\phi dx \\
&= \varepsilon\int_\Omega [(a+b\|u_\lambda\|^2)\nabla u_\lambda\nabla\phi - \lambda f(u_\lambda)\phi - \mu u_\lambda^{-\gamma}\phi]dx - \varepsilon(a+b\|u_\lambda\|^2)\int_{u_\lambda+\varepsilon\phi<0}\nabla u_\lambda\nabla\phi dx.
\end{aligned}$$

Since the measure of the domain of integration $|\{u_\lambda + \varepsilon\phi < 0\}| \to 0$ as $\varepsilon \to 0^+$, it follows that

$$\lim_{\varepsilon\to 0^+}\int_{u_\lambda+\varepsilon\phi<0}\nabla u_\lambda\nabla\phi dx = 0.$$

Therefore, dividing by $\varepsilon > 0$ and passing to the limit as $\varepsilon \to 0^+$ in (2.10), one sees that

$$0 \le (a+b\|u_\lambda\|^2)\int_\Omega \nabla u_\lambda\nabla\phi dx - \lambda\int_\Omega f(u_\lambda)\phi dx - \int_\Omega u_\lambda^{-\gamma}\phi dx.$$

By the arbitrariness of ϕ, this inequality also holds for $-\phi$, i.e.

$$(a+b\|u_\lambda\|^2)\int_\Omega \nabla u_\lambda\nabla\phi dx - \lambda\int_\Omega f(u_\lambda)\phi dx - \int_\Omega u_\lambda^{-\gamma}\phi dx = 0,$$

for every $\phi \in H_0^1(\Omega)$, which implies that u_λ is a solution of Eq. (1.2). □

3 The weak singular case: $0 < \gamma < 1$

Lemma 3.1. *If $u \in H_0^1(\Omega)$ is a global minimum of I_λ for some positive λ,then*

(a_1) $u > 0$ a.e. in Ω;

(a_2) $u^{-\gamma}\varphi \in L^1$, $\forall\, \varphi \in H_0^1(\Omega)$.

Proof. (a_1) Since u is a global minimum of I_λ, for $t \in [0,1]$, one has

$$\begin{aligned}
0 &\le \frac{I_\lambda(u+tu_-) - I_\lambda(u)}{t} \\
&= a\int_\Omega \nabla u\nabla u_- dx + b\|u\|^2\int_\Omega \nabla u\nabla u_- dx + \frac{a}{2}t\|u_-\|^2 + bt\|u_-\|^4 \\
&\quad + \frac{b}{2}t\|u\|^2\|u_-\|^2 + bt^2\|u_-\|^2\int_\Omega \nabla u\nabla u_- dx + \frac{b}{4}t^3\|u_-\|^3 \qquad (3.1)\\
&\quad - \lambda\int_\Omega \frac{F(u+tu_-) - F(u)}{t}dx - \frac{1}{1-\gamma}\int_\Omega \frac{((u+tu_-)^+)^{1-\gamma} - (u^+)^{1-\gamma}}{t}dx.
\end{aligned}$$

Notice that

$$\int_\Omega \frac{F(u+tu_-)-F(u)}{t}dx = \int_{u<0}\frac{F((1-t)u)-F(u)}{t}dx = -f(0)\int_{u<0} udx \geq 0, \tag{3.2}$$

$$\int_\Omega \frac{((u+tu_-)^+)^{1-\gamma}-(u^+)^{1-\gamma}}{t}dx = 0. \tag{3.3}$$

Combine (3.1), (3.2) and (3.3), we have

$$0 \leq -(a+b\|u\|^2)\|u_-\|^2 + \frac{a}{2}t\|u_-\|^2 + bt\|u_-\|^4 + \frac{b}{2}t\|u\|^2\|u_-\|^2 - bt^2\|u_-\|^4 + \frac{b}{4}t^3\|u_-\|^3.$$

As $t \to 0^+$, we obtain $\|u_-\|^2 \leq 0$, which implies that $u \geq 0$ a.e. in Ω.

Now we assume that there exists a set $A \subset \Omega$ of positive measure such that $u = 0$ for every $x \in A$. Thus we can choose a positive function $\varphi \in C_0^1(\bar{\Omega})$ such that for $t \in [0,1]$, we have $(u(x)+t\varphi(x))^{1-\gamma} \geq u(x)^{1-\gamma}$ a.e. in Ω. Notice that

$$\begin{aligned}\int_\Omega \frac{(u+t\varphi)^{1-\gamma}-u^{1-\gamma}}{t}dx &= \left(\int_{\Omega\setminus A}+\int_A\right)\frac{(u+t\varphi)^{1-\gamma}-u^{1-\gamma}}{t}dx \\ &\geq \int_A \frac{(u+t\varphi)^{1-\gamma}-u^{1-\gamma}}{t}dx = t^{-\gamma}\int_A \varphi^{1-\gamma}dx.\end{aligned}$$

From (3.1), one has

$$\begin{aligned} 0 &\leq \frac{I_\lambda(u+t\varphi)-I_\lambda(u)}{t} \\ &\leq a\int_\Omega \nabla u\nabla\varphi dx + b\|u\|^2\int_\Omega \nabla u\nabla\varphi dx + \frac{a}{2}t\|\varphi\|^2 + bt\|\varphi\|^4 \\ &\quad + \frac{b}{2}t\|u\|^2\|\varphi\|^2 + bt^2\|\varphi\|^2\int_\Omega \nabla u\nabla\varphi dx + \frac{b}{4}t^3\|\varphi\|^3 \\ &\quad - \lambda\int_\Omega \frac{F(u+t\varphi)-F(u)}{t}dx - \frac{1}{1-\gamma}t^{-\gamma}\int_A \varphi^{1-\gamma}dx. \end{aligned} \tag{3.4}$$

As $t \to 0^+$, we obtain

$$\begin{aligned}&\frac{a}{2}t\|\varphi\|^2 + bt\|\varphi\|^4 + \frac{b}{2}t\|u\|^2\|\varphi\|^2 + bt^2\|\varphi\|^2\int_\Omega \nabla u\nabla\varphi dx \\ &\qquad + \frac{b}{4}t^3\|\varphi\|^3 - \lambda\int_\Omega \frac{F(u+t\varphi)-F(u)}{t}dx \to -\lambda\int_\Omega f(u)\varphi,\end{aligned}$$

and

$$-\frac{1}{1-\gamma}t^{-\gamma}\int_A \varphi^{1-\gamma}dx \to -\infty,$$

we know the right-hand of (3.4) goes to $-\infty$, which is a contradiction. Thus, $u > 0$ a.e. in Ω.
(a_2) If $\varphi \in H_0^1(\Omega)$ and t_n is a sequence in $[0,1]$, define

$$h_n(x) = \frac{[(u_n(x)+t_n\varphi(x))^+]^{1-\gamma}-u(x)^{1-\gamma}}{t_n}, \quad \text{a.e. } x \in \Omega,\ n \in \mathbb{N}.$$

In order to prove (a_2), it is enough to take $\varphi \in H_0^1(\Omega)$ with $\varphi \geq 0$. Fix a decreasing sequence $t_n \subseteq [0,1]$ with $\lim_{n\to+\infty} t_n = 0$, it follows that

$$h_n(x) = \frac{(u_n(x)+t_n\varphi(x))^{1-\gamma}-u(x)^{1-\gamma}}{t_n}, \tag{3.5}$$

are measurable non-negative functions and $t_n \subseteq [0,1]$ with $\lim_{n\to+\infty} h_n(x) = (1-\gamma)u(x)^{-\gamma}\varphi(x)$ for almost $x \in \Omega$. By Fatou's lemma, one has

$$(1-\gamma)\int_\Omega u^{-\gamma}\varphi dx \leq \liminf_{n\to+\infty}\int_\Omega h_n(x)dx. \tag{3.6}$$

From (3.5) and (3.6), we complete the proof of (a_2). □

As above, we are going to construct a minimum of I_λ under our assumptions. Set

$$C_+ = \{u \in C_0^1(\bar{\Omega}) : u(x) \geq 0 \quad \forall\, x \in \Omega\},$$

has a non-empty interior given by

$$\text{int}(C^+) = \left\{u \in C^+ : u(x) > 0, \forall\, x \in \Omega,\ \frac{\partial u}{\partial n}(x) < 0, \forall\, x \in \partial\Omega\right\}.$$

Lemma 3.2. *Assume that (f_1)–(f_2) hold and let λ_* be as in (1.3). Then, for any $\lambda \in (0, \lambda_*)$, there exists $\tilde{u}_\lambda \in H_0^1(\Omega)$ with the following properties*

(b_1) *$\tilde{u}_\lambda \geq c_\mu d(x)$ in Ω for some $c_\mu > 0$ (independent of λ);*

(b_2) *$I_\lambda(\tilde{u}_\lambda) = \inf_{u\in H_0^1(\Omega)} I_\lambda(u)$;*

(b_3) *$\tilde{u}_\lambda^{-\gamma}\varphi \in L^1, \forall\, \varphi \in H_0^1(\Omega)$.*

Proof. Choose a sequence $\{\varepsilon_n\} \subset [0,1]$ with $\lim_{n\to+\infty}\varepsilon_n = 0$. We consider the perturbed problem

$$\begin{cases} -\left(a + b\displaystyle\int_\Omega |\nabla u|^2 dx\right)\Delta u = \lambda f(u) + (u+\varepsilon_n)^{-\gamma} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.7}$$

which have variational structure. Indeed, if $G_n : \mathbb{R} \to \mathbb{R}$ is defined as

$$G_n(t) = \begin{cases} \frac{(t+\varepsilon_n)^{1-\gamma}}{1-\gamma}, & t \geq 0, \\ \varepsilon_n^{-\gamma} t + \frac{\varepsilon_n^{1-\gamma}}{1-\gamma}, & t < 0, \end{cases}$$

then $G_n'(t) = (t^+ + \varepsilon_n)^{-\gamma}$ for every $t \in \mathbb{R}$. The functional $I_n : H_0^1(\Omega) \to \mathbb{R}$

$$I_n(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \lambda\int_\Omega F(u)dx - \int_\Omega G_n(u)dx$$

is well defined, sequentially weakly lower semi-continuous, coercive and Gâteaux differentiable in $H_0^1(\Omega)$. In particular,

$$I_n'(u)(\varphi) = (a + b\|u\|^2)\int_\Omega \nabla u \nabla\varphi dx - \lambda\int_\Omega f(u)\varphi dx - \int_\Omega (u^+ + \varepsilon_n)^{-\gamma}\varphi dx, \quad \forall\, u \in H_0^1(\Omega).$$

Therefore critical point of I_n turn out to be weak solution of (3.7). Let $u_n \in H_0^1(\Omega)$ such that $I_n(u_n) = \inf_{u\in H_0^1(\Omega)} I_n(u)$. From Lemma 3.1, $u_n > 0$ a.e. in Ω and is not zero, by the maximum principle, $u_n \in \text{int}(C_+)$. First, we prove that there exists a positive constant c_μ (independent n and λ), such that $u_n(x) \geq c_\mu d(x)$ for every $x \in \bar{\Omega}$. Consider the semilinear problem

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases}$$

and denote by $\hat{u}$ its unique solution. By [7], we know that there exists a constant C such that $\hat{u}(x) \geq Cd(x)$, for every $x \in \bar{\Omega}$. Fix some positive number β, such that $\frac{\beta(a+b\beta^2\|\hat{u}\|^2)}{(\beta\|\hat{u}\|_\infty+\varepsilon_n)^{-\gamma}} < 1$. Then $\beta\hat{u}$ is a subsolution of (3.7). Indeed, if $\varphi \in H_0^1(\Omega), \varphi \geq 0$, then

$$\begin{aligned}
&(a + b\|\beta\hat{u}\|^2) \int_\Omega \nabla(\beta\hat{u})\nabla\varphi dx - \lambda \int_\Omega f(\beta\hat{u})\varphi dx - \int_\Omega (\beta\hat{u} + \varepsilon_n)^{-\gamma}\varphi dx \\
&\leq (a + b\beta^2\|\hat{u}\|^2) \int_\Omega \beta\varphi dx - \int_\Omega (\beta\|\hat{u}\|_\infty + \varepsilon_n)^{-\gamma}\varphi dx \\
&= \int_\Omega [\beta(a + b\beta^2\|\hat{u}\|^2) - (\beta\|\hat{u}\|_\infty + \varepsilon_n)^{-\gamma}]\varphi dx \leq 0.
\end{aligned}$$

We have u_n is a supersolution of (3.7). Since the map $t \to \lambda\frac{f(t)}{t} + \frac{(t+\varepsilon_n)^{-\gamma}}{t}$ is decreasing in $(0, +\infty)$, then, $u_n \geq \beta\hat{u}$, hence, we obtain

$$u_n \geq c_\mu d(x). \tag{3.8}$$

Since u_n is a critical point of I_n, one has

$$(a + b\|u_n\|^2) \int_\Omega \nabla u_n \nabla\varphi dx = \lambda \int_\Omega f(u_n)\varphi dx + \int_\Omega (u_n + \varepsilon_n)^{-\gamma}\varphi dx, \tag{3.9}$$

for every $\varphi \in H_0^1(\Omega)$. Taking the test function $\varphi = u_n$, by (f_1), we obtain that

$$\begin{aligned}
(a + b\|u_n\|^2)\|u_n\|^2 &= \lambda \int_\Omega f(u_n)u_n dx + \int_\Omega (u_n + \varepsilon_n)^{-\gamma}u_n dx \\
&\leq \lambda \int_\Omega (\theta u_n^2 + C_5 u_n)dx + \int_\Omega u_n^{1-\gamma}dx \\
&\leq \frac{\lambda\theta}{\delta_1}\|u_n\|^2 + \lambda C_6\|u_n\| + C_7\|u_n\|^{1-\gamma}.
\end{aligned}$$

If $\|u_n\| \to +\infty$, as $n \to +\infty$, from above inequality, we have $0 < b \leq 0$, which is a contradiction. Thus, u_n is bounded in $H_0^1(\Omega)$. Therefore, there exist a subsequence (still denoted by u_n) and $u_0 \in H_0^1$ such that

$$\begin{cases} u_n \rightharpoonup u_0 & \text{in } H_0^1(\Omega), \\ u_n \to u_0 & \text{in } L^p(\Omega), \quad p \in (0, 6), \\ u_n \to u_0 & \text{a.e. in } \Omega. \end{cases} \tag{3.10}$$

Now we prove the result with $\tilde{u}_\lambda = u_0$.

(b_1) It follows from (3.8). In particular, $u_0 > 0$ a.e. in Ω

(b_2) From (3.10), we have

$$\|u_0\|^2 \leq \liminf_{n\to\infty} \|u_n\|^2,$$

$$\int_\Omega F(u_n)dx \to \int_\Omega F(u_0)dx,$$

$$\int_\Omega G_n(u_n)dx \to \frac{1}{1-\gamma} \int_\Omega u_0^{1-\gamma}dx.$$

Thus we can deduce that $I_\lambda(u_0) \leq \liminf_{n\to\infty} I_n(u_n)$. Since u_n is a global minimum of I_n, we have

$$I_n(u_n) \leq I_n(|u|), \quad \forall\, u \in H_0^1(\Omega). \tag{3.11}$$

By passing to the limit as $n \to +\infty$, it gives

$$\begin{aligned} I_n(|u|) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \lambda \int_\Omega F(|u|)dx - \frac{1}{1-\gamma}\int_\Omega (|u|+\varepsilon_n)^{1-\gamma}dx \\ &\leq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \lambda \int_\Omega F(|u|)dx - \frac{1}{1-\gamma}\int_\Omega |u|^{1-\gamma}dx = I_\lambda(|u|), \end{aligned} \tag{3.12}$$

and

$$\begin{aligned} I_\lambda(|u|) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \lambda \int_{\Omega_+} F(u)dx - \lambda \int_{\Omega_-} F(-u)dx \\ &\quad - \frac{1}{1-\gamma}\int_\Omega (u^+)^{1-\gamma}dx - \frac{1}{1-\gamma}\int_\Omega (u_-)^{1-\gamma}dx \\ &\leq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \lambda \int_{\Omega_+} F(u)dx - \frac{1}{1-\gamma}\int_\Omega (u^+)^{1-\gamma}dx \\ &\leq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \lambda \int_{\Omega_+} F(u)dx - \lambda f(0)\int_{\Omega_-} udx - \frac{\int_\Omega (u^+)^{1-\gamma}dx}{1-\gamma} = I_\lambda(u). \end{aligned} \tag{3.13}$$

By putting together (3.11), (3.12) and (3.13), we obtain that

$$I_\lambda(u_0) \leq \liminf_{n\to\infty} I_n(u_n) \leq \liminf_{n\to\infty} I_n(|u_n|) \leq I_\lambda(|u|) \leq I_\lambda(u).$$

(b_3) Since u_0 is a global minimum of I_λ, from Lemma 3.1-(a_2), we obtain that $\tilde{u}_0^{-\gamma}\varphi \in L^1, \forall\, \varphi \in H_0^1(\Omega)$. □

Proof of the Theorem 1.2. Choose $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, from (3.11) we get

$$\int_\Omega \nabla u_n \nabla \varphi dx \to \int_\Omega \nabla u_0 \nabla \varphi dx,$$

$$\int_\Omega f(u_n)\varphi dx \to \int_\Omega f(u_0)\varphi dx,$$

$$(u_n+\varepsilon_n)^{-\gamma}\varphi \to u_0^{-\gamma}\varphi \quad \text{a.e. in } \Omega.$$

By (3.8), one has

$$|(u_n+\varepsilon_n)^{-\gamma}\varphi| \leq u_n^{-\gamma}|\varphi| \leq c_\mu^{-\gamma} d(x)^{-\gamma}\|\varphi\|_\infty \in L^1(\Omega).$$

By the Lebesgue dominated convergence theorem, we have

$$\int_\Omega (u_n+\varepsilon_n)^{-\gamma}\varphi dx \to \int_\Omega u_0^{-\gamma}\varphi dx.$$

Thus, according to (3.9), we deduce that

$$(a+b\|u_0\|^2)\int_\Omega \nabla u_0 \nabla \varphi dx = \lambda \int_\Omega f(u_0)\varphi dx + \int_\Omega u_0^{-\gamma}\varphi dx. \tag{3.14}$$

Let $0 \leq \varphi \in H_0^1(\Omega)$ and fix $\varepsilon > 0$, taking the test function $\varphi_\varepsilon = \frac{\varphi}{1+\varepsilon\varphi} \in H_0^1(\Omega) \cap L^\infty(\Omega)$, we have

$$(a+b\|u_0\|^2)\int_\Omega \nabla u_0 \frac{\nabla\varphi}{(1+\varepsilon\varphi)^2}dx = \lambda \int_\Omega f(u_0)\frac{\varphi}{1+\varepsilon\varphi}dx + \int_\Omega u_0^{-\gamma}\frac{\varphi}{1+\varepsilon\varphi}dx.$$

By the Lebesgue dominated convergence theorem and Lemma 3.2, as $\varepsilon \to 0^+$, we obtain

$$\int_\Omega \nabla u_0 \frac{\nabla \varphi}{(1+\varepsilon\varphi)^2} dx \to \int_\Omega \nabla u_0 \nabla \varphi dx,$$
$$\int_\Omega u_0^{-\gamma} \frac{\varphi}{1+\varepsilon\varphi} dx \to \int_\Omega u_0^{-\gamma} \varphi dx,$$
$$\int_\Omega f(u_0) \frac{\varphi}{1+\varepsilon\varphi} dx \to \int_\Omega f(u_0) \varphi dx,$$

which imply equality (3.14). Therefore, the result of theorem 1.2 follows at once, by recalling that for any $\varphi \in H_0^1(\Omega)$, $\varphi = \varphi^+ - \varphi_-$. □

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